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TO

MARJORIE

PREFACE.

In many universities and other places of instruction Logic is a compulsory subject. In consequence there are two classes of students whose needs require to be considered. There are firstly the students—and these are the vast majority—who propose to study the subject but for a single year. I have tried in this book to provide for them a rapid survey of the subject, to indicate some of the points of view from which it may be studied, to outline what is fairly stable in its systematic doctrine and to introduce its problems. Even for the second class of students, those who intend to prolong their studies, there is much to be said for a brief preliminary survey of the field before proceeding to details. The Bibliographical Notes at the conclusion of every chapter are intended mainly for these students.

To the student engaged in private study or who lacks the immediate guidance of a teacher a hint may be given on how to read this book. A fairly continuous train of argument (a summary of which is given on p. ix) connects the various chapters. The first reading, therefore, should be rapid. It is desirable not to linger over difficulties. This is the more important since the book contains a certain amount of deliberate repetition. Topics briefly discussed in an earlier chapter are apt to arise again in a new connection, and rough provisional statements receive correction later.

It will be seen from the synopsis that the two halves of the book broadly correspond to the traditional distinction between Deduction and Induction. The student chiefly interested in the latter may commence at Chapter XI., or better, proceed to this chapter after reading to the end of Chapter III. Chapters IV. to X. may without loss be left for subsequent perusal.

Though I have tried to write an elementary manual and to avoid obfuscation I have made no endeavour to conceal the fact that Logic is a difficult subject, and one that is very controversial. A course on Logic, however brief, should provide a mental discipline. But I am well satisfied that the required discipline is supplied not so much by the traditional ' exercises ' as by acquaintance with, and thought about, genuine logical problems. I have, therefore, tried above all to introduce the problems.

There is one kind of difficulty that is more apparent than real. Students are commonly alarmed by modern logical symbols. The difficulty here arises from their unfamiliarity; but with a little cool thought the strangeness of these symbols quickly disappears. In any case they are introduced into this book only in order to explain them and to illustrate their use. They are not employed as a means of exposition. Nothing is said in symbols that is not also said in prose.

My sources of indebtedness will be sufficiently indicated by the bibliographical notes. Though the book will not, I trust, exhibit a partisan attachment to a ' school ', it will be clear that it owes much to the works and the teaching of W. E. Johnson, Bertrand Russell, Professor G. E. Moore, Mr. J. M. Keynes and Dr. C. D. Broad.

A more immediate debt is owing to my colleagues at St. Andrews, from whom I have profited greatly in discussion—to Professor G. F. Stout who on so many critical logical questions has thought so much and published, alas, so little, and to Mr. John Wisdom, the author of the articles in *Mind* on Logical Constructions.

My thanks are also due to Dr. L. S. Stebbing, author of ' A Modern Introduction to Logic ', who has read the whole work in typescript and offered most valuable suggestions.

I must also express my gratitude to Mr. G. A. Paul for his generous assistance in the correction of the proofs.

C. A. MACE

ST. ANDREWS, *July*, 1933.

THE ARGUMENT.

CHAPTERS I. to III. are concerned with very general matters. A variety of topics are introduced to which we return at later stages in the exposition. In Chapter I. four 'approaches' are distinguished but are found to converge on a single topic—the theory of inference. Chapter II. in consequence is a somewhat psychological discussion of inferential thought. Inference consists in passing in a certain way from one judgment to another. 'Judgments' are concerned with 'Propositions'; and propositions if true correspond with 'Facts'. Judgments, propositions and facts are therefore subjected to a provisional analysis in the following chapter.

Chapter IV. introduces certain elementary technicalities. It is concerned in the ways in which propositions can be combined. The general theory of the compound proposition belongs to the subject of Chapter IX., but is introduced here in order to facilitate the exposition of certain branches of traditional theory.

Chapters V. to VIII. review in very brief outline the doctrines of traditional formal logic—the doctrines of the four categorical forms, of the relations of opposition, of immediate and mediate inference.

In Chapters IX. and X. we are occupied with some contemporary developments. Traditional doctrine has been criticized from various points of view (corresponding to those distinguished in Chapter I.), and in these two chapters we are concerned with some of the attempts to remedy its defects. Chapter IX. contains a brief introduction to the generalization of logic as represented in the system of the *Principia Mathematica*. Chapter X. is concerned with the developments of certain parts of traditional formal logic into the theory of statistics, developments which give importance to what is apt otherwise to be regarded as a trivial and barren body of theory.

It is also a chapter of transition. Having its roots in traditional formal logic it also points forward to the theory of induction. With the theory of Induction the remainder of the book is concerned.

The general conclusion of Chapter XI., concerned with the nature of scientific thought, is that the scientist in the pursuit of his inquiries commences with the observation of particular matters of fact and ends by a systematic formulation of a set of generalizations. Hence in Chapter XII. the methods by which generalizations may be established are reviewed. The inquiry enables us to make a provisional classification of the main types of Induction. Subsequent discussion is throughout directed very largely with reference to a single problem—that of the logical foundations of inductive inference. Representative doctrines are considered not only for their intrinsic interest but with special reference to their contribution to the solution of this central problem. In Chapter XIII. one of Russell's doctrines is considered in detail as providing what is perhaps the simplest of the solutions that has ever been propounded. Chapters XIV. and XV. are devoted mainly to the more elaborate contribution of John Stuart Mill, Chapter XIV. being concerned with the general theory and Chapter XV. with the underlying principles of the Experimental Methods. In Chapter XVI. the reformulation of the principles of Demonstrative Induction suggested by Johnson are discu this discussion bringing into prominence the fundamental importance of the conception of functional dependence. The principles of reasoning by analogy are treated in Chapter XVII. Here too, the discussion is concerned not merely with the subject for its own sake but also for its bearings upon the general theory of induction. In a concluding Chapter (XVIII.) the problem is reviewed as a whole. Here we follow Dr. C. D. Broad in distinguishing ontological, epistemological and more purely logical considerations relevant to the question.

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CHAPTER I.

APPROACHES.

§ 1. The Common-sense Approach.

EVERYBODY thinks. Thinking is as fundamental a function of the mind as digestion is of the body. And as, when the digestion becomes disordered, we attend to the nature and to the conditions of this process, so, when thinking goes astray, we attend to thought and to its conditions. In this way we are led to the study of Logic.

The conception of Logic as the study of the principles of correct thinking is not the only fruitful conception, it is not the most important, but it is one that is very widespread, and perhaps it is the best of all to begin with.

This approach to Logic emphasizes one characteristic which is commonly expressed by saying that it is a 'normative', as opposed to a 'positive', science. Roughly, we may say,
e normative sciences deal with things at their best, or as con-
ing to certain principles of validity or value, whilst the
tive sciences deal with them as they actually are, and without
desire to criticize them. Logic is therefore said to be the
ative science of thought as contrasted with Psychology,
h deals with thought as it actually proceeds with all its
ties and imperfections. Similarly, Ethics is a normative
e so far as it deals with the principles of good behaviour,
stands contrasted with such positive sciences as Anthro-
y, and again Psychology, which deal with cond
l to its correctness. Æsthetics, too, is defined
e science concerned with the colours, shape
properties which objects must possess in
iful or to rank as works of art. Here the nat
ome branches of mathematics stand opposed as
properties generally without regard to their wo
he distinction between the normative and t

sciences has some importance as signifying that most things can be studied either from an evaluating or from a 'matter of fact' point of view. We may contemplate things with the object of passing a critical judgment on their value in some respect, or we may contemplate them merely with the desire to understand them. So, corresponding to any positive science, there might be one or more normative sciences according to the different kinds of evaluation of which the subject-matter admits. We may criticize the thought expressed in Plato's dialogues as consistent or inconsistent, as displaying or lacking formal beauty, as producing edification or as corrupting the youth of Athens. On the other hand, we may be merely curious as to why he thought as he did. Accordingly, our treatment would be described as logical, as æsthetic, as ethical, or, in the last case, as being merely psychological.

As signifying, then, that in studying Logic we are concerned with thought from a certain critical standpoint the science may be described as normative. But if we think of norms as ideals and of ideals as imaginary or unreal, or as standing opposed to fact, the conception is likely to prove seriously misleading. The norms or principles of valid thought are as positive and as matter of fact as those of mathematics. It is a mistake to oppose the normative to the positive sciences if this is meant to imply a difference of subject-matter. The distinction is largely one of the uses to which our studies will be put, leading at most to difference of emphasis or selection within a common field.

Given, then, that we are going to study thought with a
to improving our technique of thinking, and in the ho
avoiding fallacies and pitfalls, what are the problems that

Approaching the question in the manner of practical co
sense we find a suitable starting-point in the reflection th
the general purposes of life we require knowledge. A goo
of such knowledge we find ready to hand from two so
F v, there is the knowledge yielded by our senses. Fro
re supplied, primarily, with information with
nt immediate environment. Not far remove
owledge supplied by memory with regard t
past environments. This information is lo
certainty and reliability, but provisionally we
nory as preserving the knowledge gained th

The second source of immediate knowledge is that which is contained in our primitive stock of self-evident beliefs. Common sense is satisfied that we not only know that which we perceive and remember, but that we also have 'intuitions' of certain general truths, such as, that any two things which are equal to a third thing must be equal to one another. To begin with, we are justified in treating this as an independent source of knowledge, because it is by no means clear how it could have been derived from anything else.

In everyday thought no serious doubts arise with regard to the beliefs which are based upon sense perception or which appear to be self-evident. The first difficulties are encountered in the attempt to extend this knowledge.

At first sight it might seem that there are two distinct and independent ways in which our knowledge may be extended. We may increase it by accepting the testimony of other people, and we may increase it by drawing inferences from what we already know.

Testimony, however, does not provide us with an independent fund of information, for the testimony must either be a report of what the witness finds to be self-evident, or of what he has observed, or of some inference he has drawn.

In the first case testimony is unnecessary. What is self-evident to another we expect to find self-evident to ourselves. The statement of an axiom in a book on mathematics, for example, is not given us for our information; it merely draws attention to something that we already know, or to what we might have known had we turned our thoughts in the appropriate direction.

If the witness reports what his senses have revealed he may certainly tell us something that we previously did not know, but this is not an independent source of knowledge. So far as he is concerned, it is again sense perception. For us, it is a question whether, under the circumstances, from the fact that he asserts that he perceived that so and so was the case, we are justified in inferring that it was really the case. It is thus a special case of inference.

If it is an inference which is reported then again the testimony is unnecessary. Given the data on which the inference is based we have then to consider the validity of the inference in itself, quite irrespective of whom it may be drawn by.

It is therefore almost entirely with regard to the inferential process that the difficulties arise. Here it is that thought most obviously is liable to go astray. The practical common-sense approach to Logic thus leads us to the conclusion that what we have chiefly to consider are the principles of inference. We have to examine the conditions under which we can securely advance to new truths from those with which we begin.

There are, it is true, a number of subsidiary questions. Difficulties arise not only with regard to the passage from data to conclusion but also with regard to the ways in which the data may be expressed. We often do not know if an inference is permissible because we are not clear as to what a given statement means. We are led to consider the ways in which facts should be expressed in order to facilitate the inferential process. We are led to consider many other things as well, but these matters are all subsidiary to a central objective—the study of inference with a view to extending knowledge and avoiding error.

§ 2. Logic and Scientific Thought.

So far we have been concerned with the problems that arise in the everyday thinking of the practical man who is not a thinking specialist. There are, however, thinking specialists—men and women whose function it is to extend and refine human knowledge for the convenience of the practical man. This service is performed by the scientist.

Scientific thought is not fundamentally different from that which we employ in everyday affairs. Consequently, Logic considered as the study of scientific thinking does not differ in any very important way from the study of thought in general. The difference is that it raises the same kind of questions in somewhat specialized forms, and in subsidiary matters some of the questions are peculiarly its own.

This will become clear if we try to define, in a preliminary way, the difference between the two kinds of thinking. In outline, scientific thought is a development of everyday thought. It is a development in several ways.

In the first place scientific thinking aims at much greater completeness. In the beginning, man's interest in plants, for example, was probably restricted to the attempt to distinguish the edible or otherwise useful species from the poisonous. Even the amateur naturalist has a limited purpose in his observation. Scientific botany, on the other hand, aims at complete knowledge of all vegetable species. So generally; practical everyday interests are satisfied by more or less selected or restricted items of information. Scientific inquiry, on the other hand, will find no natural termination until we can answer any question which might conceivably be raised.

A second characteristic of scientific thought is its regard for *precision*. The development of scientific knowledge consists not only in an increase in the number of separate facts known, but also in what is already known in the rough becoming more clearly defined.

Our knowledge of the properties of things becomes more precise in two ways, by analysis and by quantification.

It is a curious fact that we can recognize a thing as being of a certain sort without seeing clearly what the properties are in virtue of which it is the sort of thing we recognize it to be. We know, for instance, if we are sharp, when a man is making a joke, but what a joke *is* it is difficult to say. Our primitive knowledge of things is not analytic; and one way in which science attains precision is by analysis.

Another curious fact—at least a fact that would be curious were it not so familiar—is that we can see that something has a certain property without seeing how much of that property it has. We can see, similarly, that something has a large number of clearly visible parts without seeing how many parts there are. For everyday purposes it may be enough to know that lead is heavy and wood is light, or that one country is more densely populated and healthier than another; but an important advance is made when we can define the former difference in terms of specific gravity and the latter in statistical formulæ. Measurement and the use of determinate quantitative concepts are thus among the most important means by which precision is secured.

The value of quantification lies not only in the fact that precise knowledge is desirable in itself. The extension of knowledge by inferential thought is a consideration of at least equal importance. If we know merely that some of the money in the till is counterfeit, and some is in the form of Bank of England notes, we still do not know if the notes are genuine. But if we know that 80 per cent. of the money is counterfeit and 50 per cent. consists in notes, we know that at least 30 per cent. of the money is in the form of counterfeit notes. This is a trivial example, but if an imposing demonstration were required we might consider the enormous extension of the possibilities of inference provided by mathematics. It is chiefly for this reason that the special sciences tend to assume a quantitative form.

The third characteristic of scientific knowledge is its *generality.* In the common walks of life man desires knowledge chiefly as a means to action, and for action we need particular pieces of information about particular things. We may want to know, for example, whether *this* succulent looking berry is really good to eat, or what *that* incalculable friend of ours will do in a certain situation which is just about to arise. We desire to be well informed in the sense of possessing knowledge which will be useful in practical emergencies. At the prescientific level thought is mainly concerned with the particularities of life and not with general truths.

Nevertheless, the desire to be practically well informed has led man to acquire an immense amount of general knowledge, for generalizations are easily remembered and readily applied. We answer particular questions about particular things by applying a general rule to the concrete situation. This unfamiliar fruit is probably edible, we argue, because it has a close resemblance to a raspberry, and raspberries are *always* good to eat. We expect our friend, in the coming critical situation, to do something sensational, because with all his incalculability he is *always* emotional.

The mere extension of knowledge by continued observation of particular cases might in itself make general knowledge possible, but it is the possibility of application which makes it so important. Application is a species of inference. Scientific knowledge therefore tends to take the form of generalisations or ' laws ', laws from which, as required, inferences may be drawn.

A fourth characteristic of science, which it is worth noticing at the outset, is its *systematic form.* So far as our knowledge is the result of casual experience and hearsay it amounts to little more than an unordered mass of scraps of information. When we study matters scientifically we try to get at the facts by a systematic procedure and to arrange them when discovered in an orderly way.

Consider first what is involved in the systematic arrangement of scientific knowledge. Facts, of course, can be grouped or ordered in a number of different ways. They can be arranged historically. We might describe as historical sciences all the departments of study in which we endeavour to think of events in the order in which they occurred. Similarly, we might describe as geographical or topographical sciences studies in which we attempt to ascertain how things are related in space. In this broad sense some parts of astronomy would constitute a geographical science. Of course the two sorts of science are not independent. We can study the ordinary Geography of the Earth without reference to time only because islands, continents and rivers stay in the same places for considerable periods.[1]

Another way in which knowledge can be ordered is by grouping together all the facts which concern things of the same sort, ignoring the differences between individual members of the class we are considering. That is to say, instead of describing the characteristics of a single individual person or thing, such as Charles I. or the moon, we may determine the characteristics common to members of the human race, or those common to heavenly bodies. Where there is a large class divisible into clearly marked species, such as the class of animals or the class of plants, we can assemble a vast amount of systematic knowledge by ascertaining the resemblances and the differences between these species. This yields what we may call the general descriptive sciences.

The descriptive sciences thus consist of generalizations about the co-existence of properties in certain things. There are, however, also generalizations to the effect that one property is manifested in connection with another property, or that one sort of event occurs in connection with another sort of event, irrespective of whether the two properties or events are located in the same thing. In this case we say that the science systematizing these generalizations is more abstract. Physics, for example, is an abstract science, since in each of its departments properties are studied in relative isolation from the particular kinds of thing they characterize. One branch will deal with the generalizations which can be made about movements, another with those about temperature and so on. Chemistry is a concrete and descriptive science so far as it deals with the specific properties of different kinds of material substance. It becomes abstract when the chemist deals with the process of chemical combination generally, without regard to the special forms of this process in particular cases.

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[1] In this paragraph the term ' science ' is used in a very broad sense to cover any branch of systematic knowledge. Commonly, however, the name is restricted to branches of generalized knowledge.

Generalizations being presupposed, perhaps the most important way of ordering scientific knowledge consists in the arrangement of these generalizations in a hierarchy according to the degree of their generality. If we consider a simple generalization such as that *water runs down-hill* we find that it is only a special case of the much wider principle that *matter gravitates.* In the same way the special laws relating to the combination of oxygen with hydrogen may be 'subsumed', as the technical expression is, under wider generalizations concerning the interaction of atoms and molecules in general.

The fact that laws about special kinds of things can be subsumed under wider laws opens up an interesting possibility. The number of independent laws to account for the behaviour of things can be in this way considerably reduced. This suggests that it might be possible to find a quite limited number of laws in terms of which all that happens in the world could be scientifically explained. This simplification has already gone a long way in the case of physics Newton's laws of motion were a considerable simplification of the principle previously accepted in accounting for the movements of material bodies. Although these laws themselves require modification, it is confidently expected that in the end even greater simplicity will result.

Similarly, until comparatively recent times, it seemed that the laws of chemistry were a quite independent set of generalizations about material things; but with further knowledge of the structure of atoms it now appears that the laws of chemical interaction are special cases of still more general laws concerning the physical properties of the ultimate constituents of matter. In other words chemical laws are 'subsumed' under the laws of physics.

Some scientists believe, too, that it is possible to subsume under these laws the special generalization concerning the bodily processes and the behaviour of living organisms. This would seem to be the most fundamental doctrine in what is described as the 'mechanistic interpretation of life'. How far this process of simplification can be carried we do not yet know.

Summing up, therefore, we may say that science is an attempt to obtain complete, precise, generalized, and systematic knowledge of the world in which we live. In studying scientific thought, consequently, we are studying the principles of correct thought in some of their most important forms.

The principles of valid inference are again in the foreground. But instead of asking the somewhat special questions which arise in regard to everyday thought we raise these questions in much more general forms. We may be led to study Logic by reflecting upon the fallacy we committed in accepting a counterfeit coin, or wondering how a jury was misguided into reaching a faulty verdict, but in the study of scientific thought we are faced with broader problems. When are we justified in generalizing from a limited number of examples? This is the problem of Induction. What are the general rules to be conformed to in applying a generalization? The answer is largely contained in the theory of Deduction.

Subsidiary questions are likewise generalized. How can knowledge be given the maximum of precision so as to make possible the maximum of extension by means of inference? The question leads us to consider what is involved in measurement, the general principles of statistical reasoning, and the theory of definition. How can our data best be systematized? And what inferences are possible when they have been systematized by scientific classification? Here we are led to logical problems concerned with classification and division.

The systematic character of science raises further questions. We have described it as systematic in procedure and as systematic in the arrangement of its results. The distinction points to two different kinds of problems—to problems relating to the process and to those relating to the product. Some of the latter have already been briefly discussed. The questions relating to process are of a different kind.

The extension of knowledge is effected in two ways, by inference from what is already known, and by the discovery of new facts upon which further inference may be based. There is thus a certain ambiguity in the conception of *scientific method*. The expression may refer to the methods of proof or to the methods of discovery. The former are matters of inference and as such are central logical topics. The study of scientific procedure for the discovery of fresh data is a late addition to Logic, and belongs rather to the special subject of 'Methodology'. The border line between Logic proper and Methodology is by no means clearly defined, and in any case we shall find the two groups of topics to be so intimately connected as hardly to admit of separate treatment.

§ 3. Logic and the General Theory of Knowledge.

We have noticed that the man of practical common sense is satisfied in the main that the deliverances of his senses are reliable, and that in his intuitions of axiomatic truths he is also on safe grounds. Where he feels the greatest need for guidance is in the process of inference.

The same is broadly true of scientific thought. It is true that the plain man or the scientist may distrust particular observations, and that he may change his mind on the question whether this or that general truth is really axiomatic. But in neither case are there any very fundamental doubts with regard to these two sources of knowledge, nor is there great concern to establish human knowledge on any simpler basis. There are, however, circumstances in which more far-reaching doubts arise.

In the case of sense perception difficulties are felt when different senses give conflicting evidence, or when the same sense delivers a vacillating judgment. To the eye an oar in the water looks bent while the hand assures us that it has not altered its shape. Things look different through a microscope, and the more powerful the lens the greater the difference is, so that we never know when we have got at the truth. Then there are dreams and hallucinations. To the dreamer or the lunatic there is nothing in these perceptions to distinguish them from those that normally occur. Perhaps, then, all our observations contain an element of illusion?

Reflection upon these matters has led some thinkers to discredit the senses entirely and to attempt to discover a wholly independent basis for belief. It is then natural to turn to the other source of information—the intuition of self-evident general truths. This is the policy of 'Rationalism'—a name given to the doctrine that all genuine knowledge is derived from axiomatic truths our awareness of which is independent of sense perception. This view was developed mainly on the Continent, by writers such as Descartes, Spinoza, Liebniz and, later, Hegel.[1]

But even knowledge alleged to be axiomatic is not wholly free from difficulties. Sometimes reasoning from what appears to be a set of self-evident truths has been found to lead to contradiction. This means that the axioms employed are internally inconsistent, and nothing would seem to be more certain than that knowledge cannot be based upon a set of principles all of which are true whilst being mutually inconsistent.

The difficulty need not, of course, lead to wholesale doubt of axiomatic propositions. It can be overcome by rejecting one or other of the incompatible members of the original set of principles. But this points to the need for an inquiry as to the principles according to which axioms should be selected.

Moreover, even supposing this difficulty is avoided, rationalistic thinkers have found it difficult to show how by deductive reasoning from self-evident principles we can obtain all the kinds of knowledge we require. We can obtain in this way mathematical knowledge, perhaps; but it is not clear how we obtain knowledge about the things that happen in the material world. And even with regard to the knowledge so obtained there is, to many minds, something very unsatisfactory in the rationalistic view as an ultimate account. There is something mysterious in the thought that we can begin with universal truths. It seems to many thinkers more natural to suppose that what appears to be axiomatic is in some or other way a species of generalization from particular matters of fact. There is thus opposed to Rationalism the theory of *Empiricism*—the doctrine that, after all, sense experience is the ultimate source of knowledge. The members of the empiricist school of thought—mainly British philosophers such as Locke, Berkeley and Hume, and more recent writers like Mill—have endeavoured to solve the problems of sense perception, and to show how general knowledge of every kind that is attainable at all is derived through the avenue of the senses.[2]

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[1] This statement requires qualification. See the following footnote.

[2] The rough classification of philosophers here adopted requires many and serious qualifications. There is, for example, a good deal of the rationalist in Hume and more of the empiricist in Descartes. The truth would seem to be that rationalism and empiricism are merely 'ideal limits' to which a philosopher may in various degrees approximate.

It is clear that in raising these questions we are extending the field of inquiry. The problems no longer relate simply to the principles of correct inference, but to the basis of knowledge as a whole.

Problems of this kind are described as philosophical. It is difficult to define Philosophy accurately, but it is sufficient for our present purposes to say that it is the study of all such fundamental questions. Metaphysics, the study of the ultimate nature of reality, is one branch of Philosophy; Epistemology, the study of the ultimate nature and conditions of knowledge, is another. Approached through the fundamental doubts which lead to philosophical inquiry Logic might be described as a branch of Epistemology. The distinction might be expressed by saying that Epistemology is concerned with the ways of knowing in general, whilst Logic is concerned primarily with one particular way, viz. by *inferential thought*.

Logic also becomes philosophical, however, through becoming analytic. Philosophy generally is concerned with the ultimate analysis of things. Logic is therefore philosophical when we attempt anything approaching an ultimate analysis of the concepts involved in the study of inference. It is, of course, possible to study many logical questions without a detailed analysis of the concepts we employ, but analysis cannot be indefinitely postponed. Moreover, many questions with regard to scientific thought lead us to consider the ultimate nature of the world with which science is concerned. In this way 'substance' and 'causation', for example, come to be topics of logical inquiry.

§ 4. Pure Logic.

The study of thought thus leads us to pursue a number of different lines of inquiry, with varying emphasis according to the manner of our approach. This is clear not only from the foregoing general considerations, but also from the history of Logic itself. But among the topics raised inference holds the central place. No one has ever studied Logic without considering this. The theory of inference has therefore the best title to be described as the subject matter of pure Logic. Pure Logic, we may say, is the science of the principles of inference.

Now science, we have seen, is the attempt to obtain complete, precise, generalized and systematic knowledge. A special science is a special branch of knowledge which is relatively complete, precise, generalized and systematic. The special science of Logic is a relatively complete, precise, generalized and systematic account of the principles of inference.

At any particular stage in its history the subject can be only relatively complete. To begin with, logicians looked around and discovered here and there a few of the most obvious principles of inference. The principle of syllogistic reasoning was one of the first to receive attention. This amounts to the axiom that if we know something about all the members of a certain class, and then discover that certain other things are members of this class, we then know about these newly discovered members what we knew about the class as a whole. A doctor, it is reported, indiscreetly asserted at a conference that in the first family he had attended all the members were drunkards. A later arrival at the conference found occasion to mention that he and his family had been the doctor's first patients. The inference drawn by the public was guaranteed by the principle of the syllogism.

In reasoning by syllogisms a pair of suitably related propositions is required. There are also principles which govern the inferences to be drawn when we have only a single proposition to go on. These are called the principles of immediate inference.

Moving in the other direction, we find principles governing the possibilities of inference when we have sets of three, four or more propositions of a suitable kind. These principles were not all noticed at the outset, so clearly there is something to be done towards making our knowledge complete in this respect.

Again, we may endeavour to make our knowledge more precise. In Logic this is a matter of analysis rather than of measurement. Our knowledge of logical principles becomes precise in proportion as we are able to give an accurate analysis of the concepts we employ. And, as the correlative of this, we attempt to isolate the absolutely simple and ultimate notions in terms of which this analysis is made.

Generalization is one of the most important respects in which Logic has developed in recent years. Though all logical principles are general, some involve a higher degree of generality than others. The earliest formulations were of relatively narrow principles. These have subsequently been seen to be only special cases of principles of much greater scope. For example, the study of inference led logicians first of all to examine the relations which holds between two classes when one class is contained within the other. The principles of inference which were considered were thus those based upon the special relation of class inclusion. But clearly there are other relations to study, and some of the possibilities of inference involved in class-inclusions depend upon the properties of a more general type of relation of which that of class-inclusion is only a special case. The logical principles in question can then be generalized, and the narrower principles subsumed under wider.

In this way generalization leads to systematization. As in other sciences, the question arises: Which are the most general principles of all? Such ultimate principles as have been suggested have been called the Laws of Thought, though, of course, any logical principle might be so described whether it be ultimate or not.

But though an important aim of the inquiry be to discover the ultimate Laws of Thought, there is in pure Logic very little occasion to refer to the fact that people think. The reason is, we shall find, that the so-called laws of thought are really laws of things. Inference is possible only because one thing implies another. What is important in the Laws of Thought is that they involve generalizations about the implications of *things*.

It might be said in fact that there is only one law of thought, and that is quickly stated: Always think in accordance with the facts. Certain things are the case, and they are so related to other things that, in virtue of their being the case, other things are the case as well. When we infer correctly it is because we see things clearly and allow our thoughts to be guided by the lines of implication which radiate, so to speak, from the facts we contemplate. Committing a fallacy is merely going off the lines.

But whether we keep to the lines or not, whether or not we think at all, the facts themselves remain. The relations in virtue of which inference is possible do not depend on thought; and it is these relations that the pure logician studies.

Thus it is that in the pursuit of logical inquiries we tend to leave behind the simple notion that the purpose of the science is merely to help us to infer. As we become interested in the very general relations that obtain between all objects of thought our attitude becomes very much more detached. We begin, perhaps, with concern for the private and personal difficulties we encounter in our attempts to think, but we end with the disinterested contemplation of the most general and the most abstract features in the nature of things.

BIBLIOGRAPHICAL NOTES.

CHAPTER I.

Historically the most important of the classical treatises on logic written from the practical point of view is the **Port Royal Logic** (Paris, 1662. English translation by Thomas Spencer Baynes, London, 1851). Here the subject is defined as 'the art of directing reason aright in obtaining the knowledge of things'. This practical purpose has received recognition in most general treatises on the subject, but it is an end which is apt to be subordinated to scientific, philosophical or purely formal interests. **Dr. F. C. S. Schiller** has written two polemical works (**Formal Logic** and **Logic for Use**) and many polemical papers to promote the cause of a really practical logic, and **Dr. Alfred Sidgwick** has done much to present the more useful aspects of the subject in a systematic way (**Fallacies, The Process of Argument,** and **Elementary Logic**). The science of the art of thinking, however, would seem to require a psychological rather than a purely logical point of view. The possibilities of this kind of treatment are well illustrated in **R. H. Thouless, Straight and Crooked Thinking.**

The works of **Francis Bacon,** especially the **Novum Organum,** provide a suitable point of departure for a systematic study of the theory of scientific method. All contemporary discussions presuppose an acquaintance with **J. S. Mill's** celebrated **System of Logic.** Further references pertaining to this branch of the subject will be found in the bibliographical notes to Chapter XI.

No serious discussion of logical problems is possible without raising philosophical and epistemological issues, but writers in whose works these issues are prominent and explicit may be described as philosophical logicians. The description covers every form of Aristotelian Logic—by which is meant the many (and diverse) systems of doctrine which exhibit historical continuity with the teaching of Aristotle himself. The logical works of Aristotle may now be read in English in the Oxford edition, edited by W. D. Ross. **H. W. B. Joseph, Introduction to Logic** (Oxford, 1906), provides an excellent exposition.

Other standard works are **F. H. Bradley, Principles of Logic; B. Bosanquet, Logic; W. E. Johnson, Logic; C. Sigwart, Logic** (translated H. Dendy), and **J. Cook Wilson, Statement and Inference.**

Reference to works on formal and symbolic logic will be found in the bibliographies to later chapters. **J. N. Keynes, Formal Logic,** would perhaps be universally accepted as by far the best general exposition of traditional formal logic, 'a work in which' (W. E. Johnson went so far as to remark) 'the last word has been said on most of the fundamental problems of the subject.' Until recently no elementary introduction to symbolic logic was available. This deficiency, however, has been remedied by **L. S. Stebbing's, A Modern Introduction to Logic,** 1930, a work which also broadly covers the whole field of logical studies. **The Encyclopædia of the Philosophical Sciences,** Vol. I., **Logic,** contains important essays on special subjects and expresses varied points of view.

CHAPTER II.

THOUGHT AND INFERENCE.

§ 1. The Nature of Thought.

To begin at the beginning, what is thought?

A complete and final answer to this question can hardly be expected at present, but some preliminary analysis is required before we can bring the more detailed questions of Logic clearly into focus.

Thinking, we may say, is a mental process, and it is divisible into individual thoughts which pass through our minds in single file, or sometimes, perhaps, three or four abreast. This leads to the question: What is an individual thought?

An individual thought might be said to be the same as an idea, but this latter word is ambiguous. Suppose I am thinking about horses, thinking, say, that *with the growth of mechanical transport the number of horses in the world is likely to decrease.* The word ' idea ' may in the first place be taken to mean the sort of thing which is immediately present to my mind when thoughts like this occur, and which corresponds to a word like ' horse ' or ' number ' in the sentences in which my thoughts might be expressed. That is to say, the idea corresponding to the word ' horse ' may be a horse itself, or the class of horses, or a horse as it appears, or something which represents a horse in the mind, a mental picture of a horse, some property belonging to horses, something more or less resembling a horse, or something produced by it in my mind. The idea may be any of these things according to the view we take as to what is immediately present to the mind when we think about horses. But what is common to all these views is that the idea is something corresponding to a single word or at least to a phrase less than the sentence as a whole.

Contrasted with this usage is that in which my idea is what is expressed by the sentence as a whole—the whole thought *that·with·the·growth·of·mechanical·transport·the·number·of·horses·in·the·world·will·decrease*. This is probably the most common sense in which the word is employed in daily life. When we say that 'an idea has occurred' to us, we do not mean simply that some object that a word would name has been immediately present to our minds, but that something has occurred to us which would take at least a whole sentence to express.

In other words, thought involves constituents of two orders of complexity. Ideas in the first sense are simpler than ideas of the second sense. It is plausible to suppose that, just as a sentence is a unity embracing a number of words, so an idea in the second sense is a unity embracing a number of ideas in the first sense. We shall be departing very little from common usage if we describe these more complex unities as individual 'thoughts' and restrict the use of the term to this sense.

A thought, then, is the sort of constituent in our mental life which requires a complete sentence for its full expression. The logician is chiefly interested in the complex unities of this kind. The individual thought is the unit of logical doctrine. This means that the logician is concerned primarily with the properties of such unities. He studies the simpler constituent ideas only so far as this is likely to throw light upon the properties of individual thoughts and of the more complicated wholes which are composed of individual thoughts.

Complete sentences express thoughts of many different kinds, not all of which are of logical interest. 'A penny for your thoughts' is an invitation to express not only your beliefs but also your doubts, your questions, your wishes and your feelings. According to the nature of your thought, the expression will take an assertive, an interrogative, an exclamatory, an imperative or an optative form. Exclamations such as '*How magnificent*', imperatives such as '*Keep off the grass*', optative expressions such as '*Oh, to be in England!*', while they are clearly sentences, do not express thoughts of the kind with which Logic primarily deals. The thoughts which are of most logical interest are of the kind which are expressed in statements. Statements are distinctive in that they are of the appropriate form for expressing a belief, whether what is stated is believed or not.

They are also distinctive in being of the form that can represent a fact, whether whatever is represented happens to be a fact or not.[1]

Among the thoughts that are expressed in statements there are two kinds which it is of great importance to distinguish. For the purpose of having before us a clear and detailed example, let us suppose that a student is making his first acquaintance with the philosophical views of Bishop Berkeley. He is endeavouring to get clear as to the meaning and implications of the doctrine that what we regard as independent material things exist only in the mind. The following train of thought occurs :

Matter exists only in the mind of some thinking being. This must mean that this table exists only in the mind like a table that appears in a dream. So that if anything happens to stop the table being perceived it would thereby cease to be, like the table in a dream when we wake up. When I go out of the room the table ceases to exist. But this is incredible. Surely, whether I am here or not the table remains unchanged.

This is a succession of thoughts expressed in a series of statements. But it will be noticed that some of these statements express the thinker's real beliefs, for example the last '*Whether I am here or not, etc.* . . '. On the other hand, most of these statements express thoughts which he thinks in the attempt to see the author's meaning, '*This table exists only in the mind*', '*When I go out of the room the table ceases to exist*'. Neither of these statements express any belief of the student. They are thoughts in which something which might perhaps be believed is merely, as we say, 'hypothetically entertained'.

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[1] It is important to avoid confusing the thought expressed in the form of optatives, imperatives, etc., with the thought that such a thought is occurring, or with other thoughts that are likely to go with it. Thus if I ejaculate '*Oh, to be in England !*' it is presumably the case that *I wish to be in England*, but the expression '*Oh, to be in England*' does not state this. Similarly, in ejaculating this I may quite well be thinking that *To be in England is desirable*, but I am not stating this either. The difference appears to be that in what is expressed in the optative expression the element of wishfulness does not enter in the same way as it does in the statements. The optative expression is not 'about' wishing in the sense in which it is about England or in the sense in which the statements are about it. In terms of a distinction to be drawn later we may say that in the optative wishing is a subjective constituent of the thought, whilst in the thoughts stated it is objective. *Cf.* Ch. *III*.

On further reflection these two attitudes might come to be reversed. If the reader were converted what was at first merely hypothetically entertained would come to be believed, whilst what formerly was believed would be considered merely as a possible alternative.

These considerations suggest that in any thought there are two factors either of which may vary while the other remains unchanged. There is, on the one hand, the mental attitude of believing or of hypothetically entertaining, and on the other, what is believed or hypothetically entertained. We may distinguish these as, respectively, the subjective and objective factors in the thought. In the special case of thoughts expressed by statements we may speak of the whole thought as being a judgment, and the objective factor, so far as it is possible to consider it in abstraction from the mental attitude or subjective factor, as a proposition.

Judgments therefore take the two important forms of beliefs and suppositions. These perhaps are not the only forms. We may also disbelieve or doubt a proposition, but these forms of judgments probably do not differ from beliefs or suppositions in the same fundamental way as that in which the two latter differ from one another. One important form of mental activity is that in which, starting with a proposition hypothetically entertained, we endeavour 'to make up our mind about it'. In this endeavour we may pass through various phases and degrees of doubt until we reach a conclusion in the form of belief or disbelief. This is the process of judgment in the everyday sense of the term. This leads us to the analysis of thinking considered as a process, a necessary preliminary to a clear understanding of the nature of inference.

§ 2. The Process of Thought.

The process of thought, of course, takes many forms. Let us examine two or three.

Firstly, there is that very elementary type of process which consists in continued attention to an object presented to our senses or in observing some sequence of events. Suppose I am making a simple examination of some liquid in a test-tube before proceeding to the special methods of chemical analysis.

My total mental process does not consist simply of sensations of colour, taste, smell, etc. Each sensation provides the basis for a judgment: *It is green, it smells like peppermint, but the taste is not merely of peppermint*, and so on. This is a genuine process of thought, a series of thoughts each of which consists in noting that the object in question has a certain property. The way in which each judgment is based upon sensation is of logical interest since it has been supposed by some logicians to be a special case of inference. Apart from this—a point we may leave for discussion later—the development is not of the inferential kind. Each fresh step depends upon the provision of fresh data, and each involves a primitive and original belief. For, it will be noted, in this process, every judgment involves the attitude of belief.

The same is true in watching a sequence of events; and analogous trains of thought are found in the sphere of memory, as when, for example, we endeavour to recall the incidents of an eventful holiday.

The recollections of the holiday might suggest the possibilities of a novel. Memory may give way to idle reverie and the play of fancy. This is a process of a different kind. So far as the novel is of picaresque variety and the imagined events are not simply the product of necessary connections, inference, if present at all, is very much in the background.

The hero hires a caravan, he gets into difficulties in a thunderstorm, he quarrels with a farmer, and has an encounter with gipsies, and so on. Here there may be principles in accordance with which thought develops, and the implications of things may exercise a restraining influence upon the imagination, but there is little positive development by means of inference. As in the perceptual or the memory process each thought is a relatively independent item. The train differs from that of observation or recollection in that the subjective attitude is throughout one of mere supposal, not of belief.

Consider next the following. We are faced with the problem: *How did this vase on the window-sill get broken? The children? But the children have been out all day. Burglars? But the silver has not been taken, and nothing else seems to have been disturbed. The wind? But the window is closed. Suppose, however, the window had been open. It is a casement window, and had the wind veered S.W. it might have blown the window to. True, it would not have struck the vase, but the handle of the window might have* *caused something to fall against it. The position of this book suggests that it may have fallen in this way.*

We make further inquiries and proceed:

John had been reading by the open window. He left the book standing on end on the sill, just where the handle would have struck it had the window blown to. The wind changed to S.W. about half an hour after he left the room.

Therefore it must have been the wind after all.

Even so simple a train of thought as this raises points of logical interest. Here it is that we meet special forms of inference such as the syllogism.[1] We may therefore examine this argument with a view to noting how it differs from other trains of thought.

First of all, we may notice that here, as in all genuine cases of reasoning or inference, there is a clear distinction between the data and that which we discover from the data. In the example cited there are thoughts such as the following: '*The children were out all day*', '*The silver has not been taken*', '*The wind was in the S.W. while the book, the vase and the window were in such and such relative positions*'. These are the items of information by means of which we were enabled to make up our minds on various points—points which in part or in whole answer the question originally raised. '*The children did not break the vase*', '*There were no burglars to break it*'. '*It was the wind which was responsible for the mishap*'. These are our conclusions—conclusions in the very literal sense that they constitute the terminal judgments we reach on the question vexing us. This distinction is found wherever there is inference, reasoning or proof.

We note, then, that in inference there are two judgments, and that they are different. This is a point not only to note but also to emphasize. It is important because we obtain an illusion of inference when this simple condition is not in fact complied with. The illusion arises from the fact that we are commonly concerned with thought expressed in language—and two different sentences may express one and the same thought.

For example, suppose some one to argue: '*You say that smoking is bad, but I disagree. Smoking is obviously desirable because it is desired by myself, for example, and by many other people as*

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[1] Incidentally it illustrates a common type of fallacy, which later will call for analysis. The student may with advantage anticipate this discussion by discovering the fallacy for himself.

well. And as I mean by " good " simply " desirable ", I infer that it is good to smoke '.

If we quarrel with this series of statements it must be either because we consider that the speaker is using words incorrectly or because we consider that he does not really mean by them what he thinks he means. If, however, we accept his word that he does mean what he says he means, we cannot quarrel with his *argument*—because there is no argument to quarrel with. If he really means by ' is good ' precisely what he means by ' is desirable ', and if he really means by ' is desirable ' precisely what is meant by ' is desired ', then we are justified in expressing his thought by using one of these terms consistently throughout. It then reads : *Smoking is desired, because smoking is desired. Hence I infer smoking is desired.* All that we have left out of the original statement are the assertions that told us that we are justified in making these substitutions. So expressed the thought clearly involves no inference.

This point arises in what are called ' merely verbal disputes '. It also arises with regard to many traditionally recognized forms of inference. It has been held that these are not genuine inferences, but only verbal transformations. One test, then, which we have to apply in such cases is simply this : Are there really two different judgments as well as two different forms of expression ?

A second characteristic of the inferential process is that both premiss and conclusion must take the form of judgments. It is not enough that the data and what is obtained from the data should be different. They must be different *judgments.*

Beliefs acquired by inference are, of course, derivative, but not all derived beliefs are reached by means of inference. What other means there are may be left to the more general theory of knowledge to decide. It is, in part, merely a matter of convenience to restrict the application of the term ' inference ' to the process in which one judgment issues from another in a special way, but it is in conformity with common usage to do so. We may inquire later on just how close the parallel is between inference and other ways in which beliefs may be derived, and how far the various processes have characteristics in common.

For this reason we must avoid speaking loosely of drawing inferences from physical events or from material things. A detective, we tend to say, makes inferences from footprints and flakes of cigarette ash, but it is not the footprint or the flake of ash itself which constitutes his premiss. The inference is drawn from the perception or thought expressed by '*Ha! This is a footprint*' or '*This is a piece of cigarette ash*'. So, too, when we speak of drawing inferences from the facts, the facts, here, are not simply items of independent reality but objective constituents of some thought.

On the same ground we are justified in ruling out one sense at least in which it might be said that our judgments of perception are inferences from our sensations. It may be true that my belief that '*This test-tube contains a green liquid*' is based upon my having certain sensations, but we cannot say that the judgment in question is an inference from these *sensations*. At most it would be an inference from the judgment: '*I am having certain sensations*'. Whether it be such an inference or not could only be decided by a discussion of the general theory of perceptual knowledge. The principle we have here laid down does not depend upon any special theory with regard to this. It is here merely contended as a matter of convenience it is best to restrict the use of the term 'inference' to processes which begin and end with judgments.

There is a third feature of inferential thought to be noted. Not every passage of thought from one judgment to another is an inference. There must be some special kind of connection between these judgments to constitute one an inference from the other. The premiss must in some way determine the conclusion. But what is the kind of determination involved? This we shall find to be the very heart of the question with regard to inferential thought.

It is not enough to say that the first judgment is the cause of the second, though this may in fact be true. Probably there are many ways in which one judgment may be the cause of another quite apart from inference. But in inference there is some more special connection. In the example considered we find that one thought determines another thought because of some relation between the things thought about. In other words, there are certain relations between the propositions which constitute the objective contents of the judgments—relations which we express by saying that the one seems to *imply*, *involve* or *entail* the other.

Consider the case in which I am making judgments about the liquid in the test-tube. I start by judging that *it is green*. I proceed to judge that *it is extended in space*. So far as the process was of a purely perceptual kind without inference there were two independent observations. But I need not have made them independently. Having found that the liquid was green I could then have realized that it must be the case that it was extended in space. Nothing can be green without being extended, i.e. the first fact necessitates, implies, involves or entails the other. It is this kind of relation which determines the path of thought in inference.

In our reasoning concerning the broken vase we find the same relation holding. The vase could not have been broken by anyone at a remote distance from it. The children were at such a distance. These two facts together necessitate the innocence of the children in the same inevitable way as being green necessitates being extended.

Shall we say, therefore, that when the objective content of the premiss necessitates or implies the objective content of the conclusion the passage of thought from the one to the other constitutes inference? This is somewhere near the truth, but it is not quite true. It says too little in one respect and too much in another.

It says too little because it is possible for one thing to imply another without the connection being apparent. In such cases the passage of thought from one to the other is not inference. Thus it would not have been inference if I had passed from thinking that the liquid was green to thinking that it was extended, whilst overlooking the fact that one implied the other. In such a simple case it would be difficult not to see the connection, but cases in which the relation is overlooked do arise—particularly when the premisses embrace a large number of circumstances of a complicated kind. We must add that the implication must be in some way cognized.

On the other hand, we should be saying much too much if we deny that there can be inference when this necessary connection is not there. Suppose, for example, that in thinking about the properties of the liquid I became confused and think that being extended implies being green, just as being green implies being extended. Then, if I was told that the liquid was extended, I should 'jump to the conclusion' that it was green. I should be guilty of a faulty inference, but it would nevertheless be an inference. The most that we can say therefore is, not that the premiss must imply the conclusion, but simply that it is thought to, or seems to, imply the conclusion. If we say that it must really imply the conclusion we are defining only a certain kind of *valid* inference. Inference in general must be defined so as to cover fallacious inferences as well.

To meet this objection we might say that inference consists in a passage of thought from one judgment to another determined by the fact that the first is believed to imply the second. Can we take this as a definition of inference in general ?

It will be found that this definition, also, is too narrow. It covers only one kind of inference, though, we shall see, a very important kind, viz. demonstrative as opposed to problematic inference.

§ 3. Demonstrative and Problematic Inference.

The distinctive characteristic of the type of inference considered in the preceding section is not that it always yields conclusions that are absolutely certain but that its conclusions are no less certain than the premisses. If the premisses are indubitable the conclusions are indubitable ; if they have an assignable degree of probability the conclusion will have the same degree of probability. There is no loss of assurance in passing from the data to what is inferred from the data. The whole of the certitude of the premisses is transported to the conclusion. It is in this sense that the inference is said to be demonstrative.

With this we may contrast the case in which the premisses, whilst modifying our attitude to the conclusion, do not compel us to accept it with an assurance equal to that with which we accept the data. However certain the premisses may be the conclusion is always less certain.

A very large number of the inferences we make in the course of everyday thought are of this kind. The conclusions we come to are not rigidly entailed by the premisses on which they are based, but they appear to be reasonable conclusions from the facts :

No one answered the bell ; therefore, I suppose, the whole household is out.

That man deceived me once; therefore he will do it again if I allow him an opportunity.

These are genuine inferences, and they are not necessarily fallacious; but it would never be claimed that the conclusions have the certainty that pertains to the premisses. We admit that the members of the household might one and all have been ill, or have failed to have heard the bell; we admit that characters may be reformed or that deceptions are not necessarily repeated, but we feel justified in giving more credence to the conclusions we have drawn than to any of the alternative propositions that are consistent with the data. The conclusions drawn are, in any case, much more certain than would have been the case in the absence of the premisses adduced.

The same is true of the rigorous and technical procedure of the law courts. It is comparatively rare for guilt or innocence to be established with absolute certitude. It is sufficient if there be a strong rational presumption in favour of the decision of the jury.

Let us consider a case of this problematic type of inference in which the degree of certainty which the premisses confer upon the conclusion admits of some exactitude of statement.

A rumour is abroad that our friend X has been drowned at sea. The evidence, it appears, is that the vessel on which he was travelling has gone down and of the thousand passengers aboard there are only a hundred survivors. This, let us suppose, is certain. How does this information bear upon our attitude to the rumour? Obviously, we should say, the chances are ten to one that the rumour is true. In other words, we can validly infer from the data that *X is probably drowned*, and the degree of probability here admits of quantification.

But '*X is probably drowned*' is not the same proposition as '*X is drowned*'. Moreover, it is a proposition which is as certain as the premisses from which it is inferred. The question we are interested in is what is the bearing of the evidence upon the proposition '*X is drowned*'? Clearly it does not entail it, nor can we say with certainty either that the proposition is true or that it is false. The most that we can say is that our attitude to it is influenced in some way by the data given. We are strongly inclined to believe it, and this is a rational inclination.

The case illustrates the way in which premisses which do not compel us to accept a given conclusion exercise a certain constraint in its direction. This is what is meant by the 'force' of an argument. Not every argument is coercive, but even a very weak argument may exercise some force in determining a rational judgment.

Suppose we are presented, one by one, with the names of the survivors of the shipwreck. The fact that the first name on the list was not the name of X would not be a strong argument for the conclusion that he was not among the survivors, but it does slightly increase the probability that this is the case. For by the time the ninety-ninth name has been given, provided X's name is not among them, we shall be practically convinced that his life is lost. Each name imperceptibly increases the evidence for the conclusion.

The analysis of problematic inference leads to a change of emphasis in the treatment of inference in general. In dealing with demonstrative inference it is possible to stress the *sequence of the thoughts* as we pass from premiss to conclusion. In performing such an inference a new idea may enter the mind. Inference may coincide with discovery. In problematic inference, on the other hand, the salient feature is the *change of attitude* to some proposition antecedently present to thought. What is at first considered as a 'mere abstract possibility' comes to be contemplated with some measure of assurance. We shall find that even in demonstrative inference this change of attitude may occur rather than any simple movement of thought from premiss to conclusion.

In both cases the validity of the change of attitude depends upon the awareness of relations that obtain between premisses and conclusion. In the case of demonstrative inference these are relations in virtue of which the conclusion is strictly entailed. In problematic inference the relations are of the kind which may be called 'probability relations' since they confer a greater or less degree of probability upon the conclusion. Both kinds of relations may be spoken of indifferently as 'evidential relations'.

The facts of inference will then be covered by some such definitions as the following: *Inference is a process of thought in which the attitude to a given proposition, the conclusion, is determined by the consciousness of evidential relations holding* *between this proposition and other propositions employed as premisses.* The relatively vague expression 'consciousness of evidential relations' must be employed to cover faulty inferences, in which evidential relations are mistakenly supposed to be present. Apart from this case, and for the purpose of defining valid inference, it is better to speak of inference as depending upon the perception or intuition of evidential relations.

Valid inference, we might say, occurs whenever the degree of assurance that we feel in regard to a given proposition is appropriately modified by the perception of evidential relations between this given proposition and some other proposition. Where the evidential relation is one of strict entailment the 'appropriate modification' consists in attaching the same degree of assurance both to the given proposition and to the entailing proposition. Where the evidential relation confers probability only the assurance attaching to the conclusion is modified, but is less than that which attaches to the premiss.

In the foregoing discussion we have been concerned simply with the nature of inference. We have not inquired into the conditions under which an inference is valid. It is easy to see, however, that in general it depends upon two sorts of conditions. It depends firstly upon the objective character of what is thought about, i.e. upon the nature of the relations between the propositions considered. It depends also upon the relation of these propositions to the thinker. These two types of conditions have been described respectively as 'constitutive' and 'epistemic'. The first of these terms refers to the constitution of what is given for thought; the second term refers to relations of the kind with which epistemology is concerned.

§ 4. Proof and Discovery.

It may seem, as has already been noted, that in demonstrative inference proof coincides with discovery. The thought that leads us by its implications to think of a new truth at the same time proves it. It is appropriate that we should think of what implies before we think of what is implied; and this might lead us to suppose that the advances of knowledge may be secured best by trains of reasoning in which the order in which thoughts occur conforms to the logical order of things that are thought about. It might even be supposed that all that can be thought about or known is constituted in such an orderly way that a perfectly rational being might begin from some suitable starting-point and proceed by progressive inferences from one judgment to another until all that is knowable is known. The train of thought would exemplify an ideal correspondence between the order of discovery and the order of proof. It is instructive to inquire just how such a conception fails to represent ordinary scientific thought as it actually proceeds.

In the first place the development of thought in the way proposed would depend upon the possibility of selecting a certain starting-point. But granting that there be a set of propositions from which all else could be inferred, it is extremely improbable that this collection of premisses has been available at any time in the past history of thought. Thought must commence from a less complete body of data. What happens to be at hand at the commencement of any inquiry is likely to be very much more fragmentary, and something from which only a very limited number of inferences can be drawn.

Again, even the knowledge we have is in a certain way redundant. Much that we might have inferred is known independently by direct observation. Sometimes the existence of an object, or the occurrence of an event,—an eclipse for example—is deduced from prior knowledge before it is perceived; but it is as equally frequent for observation to anticipate inference.

There is a further consideration which impairs the possibility of any close parallel between the order of proof and the order of discovery. What is meant by this parallel is that if C follows logically from B, and B follows in the same way from A, then the thought of C will follow that of B in time, and B will similarly follow A. But, it is to be noted, that whilst the order of time is relatively simple and unambiguous logical order is more complex. Implications are sometimes reciprocal, A implying B and B implying A. In this case there is no single direction in which thought must proceed in order to conform to the logical order.

Nevertheless, it might still be supposed that so far as we restrict ourselves to reasoning proper, without recourse to new sense perceptions we are bound to follow one or other of the lines of implication which as a matter of fact obtain. To examine this position it is best to have before us a concrete example. Let us return then to the case of the broken vase.

The train of thought opens with a question: How did the vase get broken? This is characteristic. Thought normally commences from, and is controlled by, a question. It is only by reference to a *question* that we can decide whether a given item is logically relevant, and the question determines the lines of inference pursued.

The first step is a guess. The thinker jumps to a conclusion or possible answer to the question: *The children broke the vase.* This, too, is typical—the guessing of a conclusion, not an inference from the fact observed and calling for explanation.

Then there follows an inference, in fact, a pair of inferences:

Suppose the children broke the vase. Then they must have been there to do it.

But they were not at home. (A new datum supplied by memory, evoked by the inference.) *Therefore they did not break it.* This is the first part of the final conclusion.

Guess number two: *The burglars broke the vase.* Then follows another pair of inferences.

Suppose a burglar had been here. Then he would have taken the silver.

But the silver has not been taken. (A new datum supplied by observation.) *Therefore a burglar did not break the vase.* This is the second part of our final conclusion.

Guess number three: *The wind broke the vase.* Here the data are more complex, but the inferences may be summed up as follows:

Suppose the wind was responsible. Then such and such circumstances must be supposed.

These circumstances did as a matter of fact obtain.

Therefore the wind was the cause.

The whole process consists in three trains of inferential thought, the conclusion of each contributing, negatively and affirmatively, to an answer to the question, viz.:

The children did not break the vase.
Burglars did not break the vase.
The wind did break it.

But these conclusions, or their contradictories, are present to mind before the premisses. The first step of genuine reasoning seems to consist in an inference in which the apparent conclusion or its contradictory plays the part of a premiss. The conclusion of this inference is a judgment which in the first two cases conflicts with a known fact. Consequently the contradictory of this conclusion is taken as a premiss, and there is an inference back to the contradictory of the original premiss, i.e. to the contradictory of the original guess which now becomes a final conclusion. In the third case the inference from the hypothetically entertained conclusion leads to consequences which further observation shows to be in accordance with the facts. This group of facts is then taken as a premiss, and there is an inference back to the original conclusion—the wind broke the vase.

This analysis indicates that so far from thought following the logical order the conclusion is present to mind first and the premisses come later. It appears that we first obtain our conclusion and then look round for premisses to prove it—and this in fact appears to be normally the case.[1]

In general we shall find not only that the premisses come to mind as a matter of fact after the conclusion has been hypothetically entertained but also that it is only in the light of the proposed conclusion that the data becomes significant. Apart from the thought of the possibility of burglars the presence of the silver in the house would seem wholly irrelevant to the question at issue. Apart from the hypothesis concerning the window the direction of the wind and the presence of the book would similarly appear irrelevant. It is one of the absurdities of detective fiction that the detective unravelling the mystery is commonly credited with observing significance in facts which would certainly not merit attention apart from some hypothesis framed at a later stage in his investigations. The significant facts are not hall-marked as such. They become significant only in their bearings upon hypotheses.

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[1] But it is important to note that we can only say that the conclusion comes first and the premisses later when we are considering the two propositions in abstraction from the total judgments; ignoring, that is to say, distinctions between the mental attitudes of belief and mere supposal. The conclusion comes first, but only as a guess, or more technically as an hypothesis. The belief in the conclusion follows belief in the premisses. The sequence of the *beliefs* follows the logical order. Though the supposition of the conclusion precedes the thought of the premisses the belief in the premisses precedes the belief in the conclusion.

Summing up, therefore, we may say that proof and discovery are contrasted rather than coincident processes. It is of the nature of discovery that what is to be discovered cannot be present to mind to begin with, whereas it is of the nature of proof that what has to be proved is present to mind at the outset. What remains to be discovered cannot in its specific characteristics determine the means selected for discovery, but what requires to be proved does determine the direction of search and the selection of premisses.

The dual movement of thought from supposition to fact, and back to fact from supposition, is characteristic of scientific investigation involving the use of hypotheses. This is the case, notably, in the search for causes and explanations. As a rule, there is nothing in the nature of the fact to be explained from which its causes can be demonstratively inferred. The alternative is to "try out" some hypothesis, i.e. to make some supposition which *prima facie* would seem to account for the fact and to develop its consequences.

It is essential that the fact to be explained should follow demonstratively from the hypothesis framed. But this alone is not enough. The implications of our supposition must be in accordance not only with the fact in which we happen to be interested but with all the facts upon which it bears. Most hypotheses will imply many things which admit of being directly perceived, but which, as a matter of fact, have not been subject to observation. It is by determining in every case the accordance of the implications with the facts that the hypothesis is tested.

There is, however, an important difference between the consequences of disagreement and those of agreement with the facts. When anything implied by a hypothesis is contradicted by observed fact the reverse implication—from the fact back to the supposition—destroys the hypothesis. On the other hand, when the implications of the original supposition are one and all in accordance with the facts the reverse relation is not one of implication, but at most only one which confers probability on the hypothesis. Accordance with the facts is an argument of weaker force. The conditions to be satisfied if a hypothesis is to be fully established are more complex than those required for disproof.

By reference to the two directions of thought it is possible to understand more clearly the two contrasted directions, to which we have already referred, in which the development of a science may proceed.

A special science may start from the discovery of a few generalisations with a restricted range. From this advance may take place in two directions. The scientist may proceed to formulate narrower generalizations by applying the laws first discovered to particular classes of cases. On the other hand, he may seek for wider principles from which his original generalizations may be demonstratively inferred.

Logic, as we have previously noted, commenced with the study of certain restricted principles of inference—such as that of the syllogism. From this may be derived more restricted forms, such as that involved in the Sorites.[1] Inquiry in the other direction consists in the search for more general principles under which special forms may be deductively subsumed.[2]

These two contrasted movements correspond to the two directions or 'senses' of the relations of evidence. In the forward movement we proceed from that which implies to that which is implied. In the reverse direction of development the problem is to discover something more fundamental which implies and establishes what is already known. The ultimate ideal, of course, is to ascertain just that assemblage of propositions from which all else could be inferred.

Evidential relations may then be considered by the logician from two points of view. He may inquire into the ways in which propositions of various kinds may be established or their probability enhanced; or starting from the consideration of the various kinds of proposition he may study their implications, or more generally their evidential value in relation to different possible conclusions. The first standpoint is that of the prosecution, with a case to be proved; the second that of a jury, whose business it is to assess the value of the evidence adduced.

The two lines of inquiry are, of course, reciprocal. But whichever we pursue we shall be concerned with the proposition, its nature and constitution, and with the ways in which propositions can be combined. For evidence must be embodied in propositions, and whatever admits of proof is itself a proposition.

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[1] Cf. Ch. VIII.

[2] The same distinction, in relation to Mathematics, is discussed by Russell, 'Introduction to Mathematical Philosophy,' pp. 1-2. Cf. Ch. IX.

Our next subject of inquiry must therefore be the single proposition.

BIBLIOGRAPHICAL NOTES.

CHAPTER II.

In this chapter it has been necessary to deal all too briefly with many difficult and unsettled questions. On the nature of thought in general and of ideas authorities still disagree. The subject is practically co-extensive with that of the whole of the theory of knowledge and with a good deal of analytic psychology. **L. T. Hobhouse, The Theory of Knowledge,** may be strongly recommended. Among smaller works on the *process* of thought **John Dewey, How We Think,** will be found to be helpful and suggestive.

The nature of inference is discussed in all the principal treatises on logic. **Keynes, Formal Logic,** Part III., Chapter IX. ; **Johnson's Logic,** Part II., Chapter I. and **Stebbing, Modern Introduction to Logic,** Chapter XII., provide more detailed discussion of the points raised in this chapter. On notion of 'entailment' consult, above all, **Professor G. E. Moore's** paper on **External and Internal Relations** in the **Philosophical Studies.**

The topic of probable reasoning is resumed in Chapter X. The authority here is **J. M. Keynes, Treatise on Probability.**

CHAPTER III.

PROPOSITIONS.

§ 1. Judgments, Propositions and Facts.

WHAT, more precisely, *is* a proposition ? It has already been described as the sort of thing that can be believed, doubted or supposed, as the objective constituent of a judgment, and as the sort of thing that can be true or false—but none of these descriptions tell us what a proposition is in itself. Clearly it is a very peculiar kind of ' thing ', even if it can properly be described as a ' thing ' at all. Is it composed of the words in which it is stated ? Is it something which exists only in the mind ? Or shall we say that it is of the stuff that facts are made of, existing as these do independently of minds and of the language in which mental states are expressed ?

These questions require a more detailed examination of the proposition in relation to the *sentence*, the *judgment* and the *fact*.

The proposition has frequently been defined as ' the sentence indicative ', and this usage has certain advantages since the logician requires some word for the linguistic expression of the content of a judgment. On the other hand, there is an equally important usage in which the proposition must be clearly distinguished from the sentence. For we may say that the same proposition is brought before the mind by two different sentences. This happens when a statement is translated into a different language, or is paraphrased. Again, when a sentence is ambiguous we may say that the same sentence conveys two distinct propositions.

These considerations, however, are not conclusive objections to the view that the proposition is, at any rate, something of the same general kind as a sentence, i.e. something to be analysed in terms of words or other symbols. We might, for example, treat all the alternative symbolic expressions for the same judgment as a class and use the term ' proposition ' so that all members of this class could be said to be the same proposition, whilst only those members having similar grammatical and linguistic characteristics could be spoken of as the same sentence. On this view the proposition, though distinct from the sentence, would be an entity of the same general kind as a sentence. It would be something composed of words or symbols.

Nevertheless, this usage would be quite inconsistent with that in which we speak of the proposition as being believed or doubted, or as true or false, or as a constituent of the judgment. It is not primarily the words that are believed or doubted, or which are true or false, but that which the words express. It is chiefly for this constituent of the judgment that the term 'proposition' is required. If we need a word to designate indifferently any expression of the same proposition we may speak of the 'logical sentence' (as distinct from the grammatical sentence). More simply, we may speak of these alternative expressions as constituting the same '*statement*'.

Restricting the use of the term 'proposition' to something that is expressed by words, as distinguished from the words themselves, how is the proposition related to the judgment as a whole which is also expressed by words?

Consider the train of thought which might occur when someone is contemplating the weather prospect for the day. The sequence of his thoughts, as he watches the changing signs of the sky, might be expressed in the three following statements:

I believe *it will be fine.*
I doubt whether *it will be fine.*
I believe *there will be a storm.*

Comparing these three judgments we find that there is between the first and the second a certain similarity and a certain difference. The difference is that between believing and doubting. The similarity lies in what is believed or doubted. Comparing the first with the third there is again a certain similarity and a certain difference. This time, however, the difference is in what is believed; the believing is common to both.

The possibility of these two contrasted kinds of variation within the judgment provides the ultimate basis of the distinction between the subjective and the objective 'constituents' or factors in a judgment. The simplest way of conceiving these two kinds of variation is to suppose that the judgment is a complex in which we can discern two distinct parts each of which may be varied independently. That which appears to be an 'act' or 'attitude' of a thinker is naturally described as subjective. The other part which constitutes the 'total object' of this act or attitude, that concerning which the thinker may change his mind, will correspondingly be described as the objective factor, or as the proposition judged. But even if it be admitted that two kinds of variation in the judgment implies two distinct 'parts' it need not be held that each is capable of independent existence. Even if the propositions were capable of independent existence it is clear that an act is the sort of thing which could not so exist.

It is not necessary, however, to hold that the possibility of two kinds of variation, in respect of the nature of the 'act' and in respect of the nature of the object, implies that the judgment contains two distinct parts corresponding to these two kinds of variations. If we consider generally the ways in which the distinction between 'act' and 'object' is drawn we find two contrasted cases. When we distinguish act and object in the case of building a house, tying a knot or seeing a star, the object—the house, the knot or the star—can clearly be distinguished as a separate and independent thing which can exist independently of the act in question. When, on the other hand, the distinction is drawn in such cases as running a race, delivering a speech, singing a song, playing a game or 'feeling a creepy feeling', the object—race, speech, song, game or creepy feeling—is incapable from its very nature of existing independently. Moreover, in these latter cases it is fairly clear that the total occurrence does not consist of two 'parts'. Nevertheless, we can distinguish two sorts of variations analogous to those noted in the judgment. We can distinguish the game from the way in which it is played, the song from the way in which it is sung, and so on.

Believing a proposition, it might be said, is only a special case of thinking a thought, and in thinking a thought act and object are distinguished in the second way, not in the first. If this be so it is misleading to describe the judgment as containing two *constituents*, act and object, and misleading to describe the proposition as the objective constituent of the judgment. The distinction of 'act' and 'object' on this view is not a distinction between *constituents* at all.

Nevertheless, even on this view, the distinction between the two contrasted types of variation will remain; and we may conveniently use the term 'proposition' to express this distinction. All that we wish to say about propositions could be put into statements about judgments, but it will be convenient to speak of the 'propositions' when we are concerned with the characteristics of judgments which vary with variation in 'what is being judged' as contrasted with what we have called the 'subjective' variation. We may perhaps describe the proposition as the objective 'factor' in the judgment, provided that it be understood that a 'factor' is not necessarily a 'part' or 'constituent' of the judgment.

Although the proposition is to be distinguished from its verbal expression and defined as the objective factor in a judgment this does not warrant us in saying that it is mental in nature. A judgment is mental, but only in virtue of the factor or the characteristics referred to as subjective. The proposition is objective in the sense that it contains constituents which would seem, ultimately, to be of the same nature as the constituents of fact. We are thus led to inquire into the relations of propositions to facts.

If we consider the passage of thought from mere supposal, through hesitant belief to assured knowledge, it would seem to be clear that what in turn is supposed, believed and known is constant throughout. If first it be hypothetically entertained that the earth is round, and then believed, and then, when proofs are found, definitely known, it would be strange if the object varied with the mental attitude. On the contrary, what was at first only supposed is precisely what comes later to be known. But once it is known we can assert it to be a fact. In this sense of the term 'a fact' is nothing but a proposition which is known to be true. Consequently, we cannot explain the nature of the proposition by reference to facts in this sense. Facts, in this sense, must be explained by reference to propositions.

There is, however, another sense of the word 'fact' the discussion of which throws further light on the nature and status of propositions. An inquiry into the relations between proposition and fact may be concerned with the connections that obtain between the contents of our beliefs and the system of reality which, *prima facie*, exists in total independence of the thoughts we have concerning it, and upon which the truth or falsity of our beliefs depends.

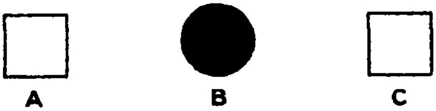
If I believe on Tuesday that the weather on the following day will be fine, and my belief is not mistaken, it is Wednesday's fine weather which constitutes the 'matter of fact' to which my belief conforms. More specifically we may say that, in this sense of the term, a fact is an abstract portion of reality. It is that portion which corresponds with, and is represented by, the words which also express the true proposition.

Since a true statement expresses both a true proposition and a fact there is a danger of confusing the proposition and the fact, and a temptation to suppose that the objective content of the belief is the fact itself. But this cannot be the case. Neither Wednesday's fine weather as a whole, nor the abstract portion of it described by the words employed can be a constituent of a belief on Tuesday. Still more obviously must we distinguish proposition and fact when the belief is false. Nevertheless, it is reasonable to suppose that the ultimate constituents of a proposition are also the ultimate constituents of a fact. Where 'whole facts' of the required constitution are lacking there is nevertheless factual material with which the belief is concerned. Judgments involve the presence to mind of certain elements of fact. These elements which in one way constitute facts may in another way constitute propositions. It is in this sense that we may regard the ultimate material of propositions as consisting not of words, nor of purely mental 'stuff', but of the entities of which the actual world is composed.

§ 2. The Elements of Facts and Propositions.

Matters of fact, though infinitely various, would appear to be reducible to a fairly simple scheme; that is to say, to be resolvable into a few basic types. The general plan can be exhibited by reference to one or two examples.

Supposing I am comparing the three following diagrams:



Attending to the first and the second, I may judge *This is square* and *That is round*, or *This is white* and *That is black*. Employing the letters as arbitrary proper names, I could also express my judgment in the form *A is square* and *B is round*, etc. Again, I may judge *A is smaller than B*, *B is between A and C*, and so on. Such judgments it will be found are representative of a large number of judgments which can be made concerning matters of fact. That is to say, if we ignore the particular 'matter' of these and consider only their form, the forms exhibited will be found to be present in an indefinitely large range of common judgments. It is, therefore, of importance to examine these types in some little detail.

We may restrict attention to begin with to the facts expressed by the statements:

This is white (or *A is white*).
That is black (or *B is black*).

It is clear that these two facts differ in two ways. They differ, firstly, in the way represented by the difference between 'This' and 'That'; and they differ, secondly, in the way represented by the phases 'is white' and 'is black'. It is to be noted that we have here not only two differences, but two *kinds of difference*. The difference between these two kinds of difference may be expressed in various ways. We may say that the first kind of difference is merely that between the *things* we are speaking about, the second is a difference in what is being said about them. Again we may say that the terms 'A' and 'B', 'This' and 'That', signify merely that the two things are 'numerically distinct' or 'other than' each other without implying any difference in character. The second kind of difference is a difference in character. In saying *A is white and C is white* we express otherness without difference in character. In saying *B is black and B is round* we express difference in character without otherness. The statement *A is white and B is black* combines both kinds of distinction.

The matter has been expressed rather more technically in the following way. It may be supposed that the facts are in some way complex. It is therefore suggested that, in the case under consideration, each is constituted by the union of two elements, and that the elements are of two kinds which may be called respectively '*constituents*' and '*components*'. The constituents in a fact (also described as '*particulars*' or '*individuals*') are the elements in virtue of which one characterized thing might be merely numerically distinct from another, i.e. distinct without difference in character. The components of a fact are elements which can in a certain sense be 'shared' in common with things which are numerically distinct. If A is white and C is white A is other than C, but whiteness is shared by both. It is their 'common property'. Such properties in virtue of the fact that they can be 'shared' are called 'universals'. These facts, therefore (facts of the kind: A is white, B is black, etc.) are constituted by the union of the two elements, one constituent or particular, and one component or universal. Components which require to be united only with one constituent to constitute a fact are called qualities.

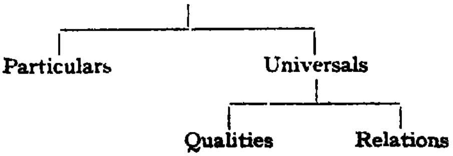
Some components require to be combined with more than one constituent in order to constitute a fact. Such components are called relations. Such a mode of union is illustrated by the fact

A is smaller than B.

For certain purposes, and in a certain sense, it is permissible to say that in this fact also we may distinguish two 'parts'—the particular represented by A and the universal represented by the phrase 'is smaller than B'. But it is evident that such an analysis is not ultimate. 'Being smaller than B' is not so simple a character as 'being white'. In some sense the component contains the particular B as a factor in its constitution. A more ultimate account of the fact is given by saying that it is constituted by three elements, the particular A, the particular B and the relation 'is smaller than'.

The relation expressed by 'is smaller than' is what is called a 'diadic' relation. It requires two 'terms'. There are, however, triadic, tetradic and more complex relations. The relations involved in the fact *B is between A and C* is triadic, inherently involving the three terms A, B and C. These multiple relations as they are called are of special importance in the analysis of judgment.

We thus obtain the following classification of the elements of fact:



If this be the correct and complete account of the matter we may say that the world divides into facts and facts ' divide ' (in another sense) into particulars, qualities and relations. These elements may be combined in various ways, but the possible modes of combination are subject to certain important restrictions. For example, a particular A can be combined with a quality *whiteness* to yield the fact *A is white*, but we cannot combine two particulars without qualities or relations to yield either facts or propositions.

Among the modes of combination which *prima facie* appear to be possible the following are some of the more important :

(1) The case in which a thing is characterized by a quality, i.e. a case in which the elements are one particular (as constituent) and one quality (as component), e.g. *This is white*, *Socrates is wise.*

(2) The case in which a relation relates two or more things, i.e. in which the elements are two or more particulars (as constituents) and one relation (as component), e.g. *This is larger than that*, *Socrates is wiser than Plato.*

If it be possible to predicate a quality of another quality or of a relation, or to assert that a relation relates two or more qualities or two or more other relations, the following cases also must be noted [1] :

(3) The case in which a quality characterizes a quality, e.g. *Pure yellow is bright*, *Pure blue is dark.*

(4) The case in which a quality characterizes a relation, e.g. *Contrast is beautiful.*

(5) The case in which a relation relates two qualities, e.g. *Wisdom is preferable to folly.*

(6) The case in which a relation relates two other relations, e.g. *Being equal to is different from implying* (since one is a symmetrical and the other an asymmetrical relation).[2]

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[1] It is, however, doubtful whether it is significant to assert that a quality characterises another quality or a relation.

[2] A symmetrical relation is one which ' works both ways '. ' Difference ' is a symmetrical relation because if A is different from B, B must be different from A. ' Being the father of ' is an *asymmetrical* relation because if A is the father of B then B cannot be the father of A. A relation is said to be *non-symmetrical* in cases such as the following : A is *the brother of* B, A implies B. Here it may or may not be the case that B is the brother of A, and that B implies A.

Interpreting these cases as propositions it thus appears that we may have propositions in which every element is a quality or every element a relation. On the other hand, we find no propositions in which every element is a particular. Hence a 'thing' or particular is sometimes defined as that which can have qualities or stand in relations, but cannot itself function in the proposition as a quality or relation. It cannot function as the predicate in the proposition. A quality or a relation, on the other hand, can function not only in its own characteristic way, but also in the way in which a particular functions.

Particulars, qualities and relations are the elements of facts and they are the elements of propositions. We may now inquire more precisely into the ways in which they enter into the constitution of these complex entities.

At first sight it might seem that there is a perfect parallelism between the two cases. Elements combined in a certain way yield a certain type of proposition. If the proposition is true there is a corresponding fact of similar or analogous constitution. Alternatively we might say, elements combined in a certain way yield a certain type of fact. If we combine them in this way in a judgment we obtain a corresponding proposition. Such a view, however, may easily be seen to be inadequate.[1]

Among the commonest types of propositions are general and indeterminate propositions. We may judge that 'Something is blue' or that 'The sky is bluish' or that 'Everything is transitory'. Now, a matter of fact in itself cannot be general or indeterminate. If 'Something is blue' is true then there is some particular thing (or things) of which 'blue' is a character. The fact cannot consist of *something or other* being blue. Similarly, although we may correctly say that the sky is bluish, the fact which makes this judgment true consists not in something having 'some or other' colouring resembling 'pure' blue, but in something having a quite specific and determinate mode of colouring. It cannot be just 'bluish'. Again, if everything is transitory there are the facts that A is transitory, B is transitory, C is transitory (and so on until we have mentioned everything), but there is not in addition to all these particular matters of fact another fact, *Everything is transitory*.

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[1] Its plausibility arises from the ambiguity of 'fact', i.e. from the sense in which a fact is a proposition known to be true. So far as *any* proposition might (theoretically) be known such a parallelism would hold. But this is not the important sense of fact.

The world of fact, therefore, consists only of particular and specific matters of fact. It consists of particulars characterized by quite specific qualities and related by quite specific relations. If we could mention all the particulars there are, stating all the specific qualities they possess and all the specific and determinate relations in which they stand to one another, we should have said all there is to say about the world.

So much for the facts. We must now consider judgments and propositions.

§ 3. Judgments and Propositions.

The important thing to notice about a judgment is that it is itself a fact. If I believe that the circle on page 39 is black then that I believe this is a fact. A judgment—when it is correct—is a fact about a fact. Hence when the judgment is correct there are then two facts which are connected in a certain intimate way. Thus let us suppose that the particular which we have named B has a specific colouring which we will represent by C. Let us suppose further that I have judged this to be the case. There are then the two facts :

(1) B is C.
(2) I judge B to be C.

Let us suppose further that these two facts are perfectly non-general and specific. We may then inquire how precisely the two facts are connected. The question turns upon the constitution of the second of the facts. On this matter two views would seem to be possible.

According to the first view the fact *that B is C* is a part of the fact *I believe that B is C*. The fact *that B is C* consists simply in the particular B (as constituent) being characterized by the quality C (as component). The second fact might then be said to consist in (i) the judger I (as one constituent), (ii) the fact that B is C (as another constituent) and (iii) the diadic relation of believing (as the component). The judgment would thus be a fact constituted by three elements, two constituents (one being itself a fact) related in a certain way by a diadic relation.

The chief objection to this account arises from the consideration that it is reasonable to suppose that a judgment in itself has the same constitution whether it be correct or incorrect. It is not plausible to suppose that a judgment has one kind of internal make-up when it is true and an entirely different internal constitution when it is false. This would of necessity be the case on the account of a judgment suggested above. The analysis could only hold, at best, of true judgments. In false judgments there is no fact to provide the second constituent of the fact of judgment.

Consequently, the second view of the matter is more satisfactory. On this account the chief component in the fact *I believe B to be C* is a multiple relation—a peculiar kind of relation which unites the elements *I*, *B* and *C* into a peculiar sort of unity—the fact that I believe B to be C. This escapes the difficulty of the first view since whether the judgment be true or false *I*, *B* and *C* and the judging relation are genuine elements of reality. The presence of this kind of relation would seem to be characteristic of mind. It is to be found in all the species of thought mentioned in the previous chapter, in perception, in memory, in mere imagination, in belief and supposal. The general formula is :

I . . . So and So to be such and such.

The empty space can be filled in by the words 'perceive', 'remember', 'imagine', 'believe', 'suppose', etc. In every case the word introduced stands for a species of this relation which unites a judger and his ideas to constitute a 'thought'. Whether a thought be true or false has nothing to do with its internal constitution, but depends upon whether there is a fact constituted by the elements of a judgment excluding only the judger and the judging relation.

So far, however, we have been concerned only with judgments of particular matters of fact—judgments which consist in someone thinking that some particular thing has some specific quality or that certain particulars are related in some specific way. We have now to consider the many and various ways in which something may be judged.

The topics here to be discussed may be introduced by reference to a classification suggested by Kant which has been adopted with varying interpretations and modifications by many logicians. Four general principles of division are here employed and under each head three classes are distinguished, resulting in the following scheme :

I. *Quantity.*	
i. Universal	(*All S is P*).
ii. Particular	(*Some S is P*).
iii. Singular	(*This is P*), (or *The S is P*).
II. *Quality.*	
i. Affirmative	(*S is P*).
ii. Negative	(*S is not P*).
iii. Infinite	(*S is non-P*).
III. *Relation.*	
i. Categorical	(*S is P*).
ii. Hypothetical	(*If S is P then Q is R*).
iii. Disjunctive	(*Either S is P or Q is R*).
IV. *Modality.*	
i. Problematic	(*S may be P*).
ii. Assertive	(*S is P*).
iii. Apodeictic	(*S must be P*).

The classification here adopted is open to objections of many kinds. Emphasis is placed upon distinctions of minor or secondary importance, and distinctions of a fundamental character are obscured. We shall find that the classification by reference to ' Quantity ' must be replaced by the more fundamental distinction between General and Non-general Propositions, that a simple dual classification according to ' Quality ' may replace Kant's threefold division, that the distinctions in respect of ' Relation ' require radical amendment, and that the distinctions in ' Modality ' admit of varied interpretations. These four topics will be discussed in turn.

General and Non-general Propositions.—It has been seen that matters of fact (in one fundamental sense of ' fact ') are always particular and determinate. A judgment regarding such a fact may be described as a non-general judgment and what is truly or falsely asserted in such a judgment may be described as a non-general proposition. It corresponds roughly to the traditional conception of a singular proposition. The statement *B is C* may be taken to express such a judgment (where

B stands for that particular presented on page 39 and C for the quite specific colouring it possesses).

A general proposition is one which involves the notion of 'some' (and derivative notions such as 'all', 'none', etc.). It will be seen that generality may enter into a proposition in two ways. We may have generality in respect of the reference to constituents and generality in respect of the components. Commencing from the judgment *This is C* or *B is C* we may proceed to assert that *Something is C*, *Everything is C* or *Nothing is C*. This involves generality in reference to the 'subject' of the proposition.

Again starting from *This is C* we may proceed to predicate more or less indeterminate characters of this subject. *This has some character*, *This is coloured*, *This is black*, etc. The predications 'is coloured', 'is black', etc., do not represent new qualities which 'this' possesses over and above the specific mode of colouring to which we have referred. They are more or less indeterminate predications of a quality. That is to say 'This is coloured' is equivalent to 'This has *some* quality of the colour kind'; and 'This is black' is equivalent to 'This has *some* shade of a blackish sort'.[1]

Such notions as 'colour', 'shape', etc., have been designated 'determinables', the specific shades of colour, and the specific forms are correspondingly described as 'determinates'.[2] It is clear that in predicating a colour of something we are in general predicating something intermediate between the extremely indeterminate predication 'coloured' and the absolutely determinate shades for which we have no names.

It is also important to observe that a proposition admits of generality in respect of each of the elements in its constitution. A simple proposition involving one particular and one quality admits of two degrees of generality. A proposition of the form 'A has the specific relation R to B' admits of three degrees of generality corresponding to (i) Something has R to B, (ii) A has R to something and (iii) A has some relation of the R sort to B. The possibility of this 'multiple generality' as we may call it was vaguely recognized in traditional logic under the name of 'multiple quantification'. The statement 'All cows are ruminants' is said to be 'quantified' in respect of its subject term since the kind of generality which attaches to this term is explicitly indicated. The form 'All cows are some ruminants' is for the same reason said to have a 'quantified predicate'. Such statements as 'All men are always truthful to some people on some subjects' is said to possess 'multiple quantification'. In this case the notion of quantification is applied more particularly to the constituents of the proposition. When in addition we recognize the generality which may attach to the components it is obvious that the number of degrees of generality in a proposition may be indefinitely great.

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[1] Since there are no names in ordinary language for absolutely specific properties all ordinary statements involve this kind of generality. We were compelled to use the arbitrary symbol 'C' for the specific character which B was asserted to possess.

[2] Cf. Johnson, 'Logic.'

It is obvious that in the Kantian classification according to 'quantity' the distinction between the so-called singular proposition and the two other forms is much more fundamental than the distinction between the 'Universal' and the 'Particular' forms. The precise character of this latter distinction and the distinction between the two types of 'singular' propositions will be discussed later.

Affirmative and Negative Propositions.—It is obvious that in regard to what have been traditionally described as distinctions in 'quality' we have only two and not three cases to consider. Kant's so-called 'infinite' form must be analysed in terms of the concepts required in distinguishing affirmative and negative propositions. The central problem here relates to the nature of negation.

Two types of view may be distinguished with regard to the nature of the difference between the two judgments expressed

by *S is P*
and *S is not P*,

when S names a particular and P stands for an absolutely specific property.

According to one view the difference resides not in the nature of what is being judged but in the attitude of mind towards what is presented for consideration. '*S is P*' expresses *the attitude of belief* towards a certain proposition, '*S is not P*' expresses a different attitude, that of *disbelief*, towards the same proposition.

On the other hand, there are considerations which favour the view that both express an attitude of belief, the difference residing in what is believed. It may in consequence be held that the difference is derivative from a more fundamental distinction between positive and negative facts. If it be the case that the sky is a specific shade of blue then in addition to this fact there is another fact that it is not of a specific shade of red. There are, in fact, on this view, an innumerable number of such negative facts. It is possible, however, to hold that the negative judgment differs from the affirmative in respect of what is judged without holding that there is a distinctive negative kind of fact. '*S is not P*', it might be said, expresses a *general* judgment, one in which generality enters into the predicate, viz. the judgment that S has *some* character *incompatible with P*. On this view '*in*compatibility' (in spite of the negative prefix) expresses a specific type of relation.[1]

Still another possibility is to be noted. It is sometimes suggested that a negative assertion always expresses a secondary proposition, i.e. a proposition about a proposition. '*S is not P*' is then interpreted as meaning that *S is P is not the case*, or *S is P is false*, or that *S is P is to be denied*, where 'denial' implies the occurrence of a peculiar mode of judging.

Simple and Compound Propositions.—The traditional classification of propositions as Categorical, Hypothetical, Disjunctive involves a confusion of two principles of division. The distinction between what is 'categorically asserted' and what is 'hypothetically entertained' is a distinction between two attitudes of mind which may be adopted towards something presented for consideration.[2] There is, however, in this classification also the distinction between simple and compound propositions.

Given relatively simple propositions these may be conjoined in various ways to yield compound propositions. The hypothetical and disjunctive forms exhibit only two of the possible ways in which such compound propositions may be constructed. But any compound judgment will exhibit both the categorical and hypothetical attitudes of mind. In the form '*If A is B then C is D*' the two simple component propositions *A is B* and *C is D* may be only hypothetically entertained, but it is categorically asserted that the acceptance of the antecedent proposition involves the acceptance of the consequent proposition.

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[1] 'Compatibility' would correspondingly become the negative conception, i.e. the absence of this relation.

[2] Cf. Ch. II., p. 18.

A compound proposition is defined as one which consists of parts which are themselves propositions, whilst a simple proposition is one which cannot be resolved into parts which are themselves propositions. The conception of a simple proposition, however, requires further analysis.

We must distinguish, in the first place, a sense of the expression 'simple proposition' in which a general proposition such as '*All men are mortal*' is simple from a sense in which only non-general propositions can be simple. Traditionally, general propositions, which cannot be analysed into a conjunction of propositions, have been regarded as simple. There are, however, in addition, *absolutely* simple propositions, propositions which, if true, would correspond to the absolutely specific and non-general facts to which we have previously referred. Such absolutely simple propositions are called 'atomic propositions'.

Compound propositions can thus be constructed either by compounding atomic propositions or by compounding general propositions. Compound propositions, all the constituent propositions in which are atomic propositions, are called *elementary* propositions.

Notice should also be taken of the difference between propositions which can be completely analysed into simpler propositions, and propositions which although containing propositions as parts are not resolvable without remainder into a conjunction of propositions. In the proposition '*I believe that B is C*' (interpreted as corresponding with an atomic fact) 'B is C' is itself a proposition, and the total proposition is to the effect that 'B is C is believed by me'. But the expression 'believed by me' does not express a separate proposition conjoined in some way with the former. It expresses rather a character predicable of propositions. The same kind of complexity is exhibited by general propositions, such as '*That the sun will rise to-morrow is probable*' or '*That ghosts exist is absurd is true*'. Such differences may be described as differences of order. Given a simple proposition (in any sense of simple) as a proposition of the first order, a proposition about this proposition may be described as a proposition of the second order. A proposition assigning some character to a second order proposition is one of the third order and so on. Second or higher order propositions may be 'atomic' in the sense defined.

Distinctions of Modality.—The term modality is employed to describe certain distinctions expressed by the phrases 'is', 'may be' and 'must be', but these expressions are extremely ambiguous.

We must distinguish cases in which 'S' (in statements of the form 'S is P', 'S may be P', etc.) is a genuine proper name from cases in which 'S' connotes a certain property. Where 'S' connotes certain properties distinctions expressed by 'is' and 'may be' may sometimes be expressed in terms of 'quantity'. The statement 'A swan must be long necked but may be black' might be intended merely to express the assertion that '*All* swans are long-necked and *some* are black'. This interpretation in terms of quantity will not be applicable where 'S' merely stands for a particular without assigning any character. Pointing to two objects we might assert: '*This is solid, but that may be hollow*'. Here the distinction between the assertoric and the problematic form corresponds to a distinction between two attitudes in the mind of the judger. Solidity is *believed* to characterize one thing, but it is *doubted* whether hollowness characterizes the other thing.

Similar considerations arise in connection with the distinction between the merely 'assertoric' and the 'apodeictic' judgment. Here, however, no distinction can be drawn between the intrinsic characters of the two propositions '*S is P*' and '*S must be P*' where 'S' is a genuine proper name. The distinction arises primarily in asserting a relation to hold between the possession of one character and the possession of another. We might quite naturally assert 'Every equilateral triangle that has been inscribed on paper is less than ten square yards in area'. This matter of fact concomitance between two characters is to be contrasted with the necessary connection asserted in judging 'Every equilateral triangle that has been inscribed on paper is equiangular'. Here something more than mere concomitance is asserted to hold between the two characters, viz. some relation between these characters in virtue of which anything which possessed the first *must have* possessed the second.

§4. The 'Subjective Factor' in the Judgment.

The foregoing discussion has confronted us with the following situation. A number of distinctions have been drawn in relation to the statements which express our thoughts. We have distinguished positive from negative assertions, general from non-general, simple from compound assertions, and so on. At first sight it was tempting to suppose that such statements differed only because they expressed, when true, correspondingly different sorts of facts—positive facts and negative facts, general and non-general facts and various kinds of facts corresponding to the various other kinds of statements. We have found it necessary, however, to reject this view. Facts though in one sense of various kinds are in another respect alike. They are all non-general and positive. The different kinds of statements, therefore, are to be distinguished not as expressing different sorts of facts but as expressing different ways in which we judge about the facts. These differences now require to be examined.

A convenient point of departure is the simple and determinate judgment—the judgment that some particular has some specific character, or that such and such particulars are related in such and such a specific way. These judgments, we have seen, have a fairly simple relation to certain corresponding facts (when the judgments are correct). If the fact consists in a particular, A, having the specific character p, then the judgment consists in A, the character p, and the thinker T being related in a peculiar way—that which we express most simply by saying that the thinker, T, believes A to be p. This triadic relation may hold between T, A and p even when there is no corresponding fact constituted by A having p; but in that case the judgment is mistaken. This lack of correspondence is what is meant by saying that the judgment is false.

This analysis of the nature of judging, however, has a curious consequence. It implies that the word 'judging' has systematic ambiguity, i.e. it stands for something different according to what kind of thing is being judged. It is evident that on this view the act of judgment must involve at least a triadic relation—a relation connecting the three terms T, A and p. But if the fact itself involves three elements the judging relation must involve five. If the thinker judges that A is a certain specific distance from B, i.e. a fact of the form A r B, then the judgment that A is *r* to B must be a fact consisting in the thinker T, the three elements A, *r* and B, and the judging relation which connects these four elements in the appropriate way. A judging relation which relates four terms must be different in a very important way from one which relates only three.

Equally fundamental differences in the nature of the judging operation call for recognition in passing from these absolutely simple judgments to negative and general judgments.

It is reasonable to suppose that thought commences in the simple judgment of the form 'S is *p*',[1] where S is a particular present to mind and *p* some specific character. From this a 'general' judgment may arise in two ways. From S is *p* the thinker may proceed to the judgment (i) *Something is p*, or to the judgment (ii) *S has some character.*

The latter type of judgment may be restricted in various ways. One restriction is illustrated by such a judgment as '*S is coloured*', which is an assertion to the effect that S has some character of a certain type. Another type of restriction is illustrated by the assertion *S has some character incompatible with p.* This is one form at least of the negative judgment *S is not p.*

From *S is p* and *S is not p* the more common types of general judgments may be simply derived. The first step consists in making these judgments about 'something' rather than about S, i.e. in judging *Something is p* or *Something is not p.* These judgments may be replaced by their contradictories. Judging that *it is not the case that something is p* we are judging that *Nothing is p.* Judging that *it is not the case that something is not p* we are judging that *Everything is p.*[2]

These extremely general assertions, in which everything whatever is asserted to possess or lack a certain character, are rarely if ever required in the course of thought. The more usual forms of universal judgment are of the type *Everything that is p is q*, and *Nothing that is p is q.* These are reached by the operation of conjoining two predications and then denying the result. Starting from the observations *This is p* and *This is q* we obtain ***something is both p and q***. The assertion: ***It is not the case that something is both p and q*** corresponds to the more familiar formula *No P is Q*. Similarly, starting from the observations *This is p* and *This is not q* we obtain by conjunction: *Something is both p and not q*. Denying this, i.e. asserting *It is not the case that something possesses the character p and lacks the character q*, is equivalent to the familiar judgment *All P's are Q's*.

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[1] In this and the following paragraphs the symbolic convention is adopted of representing particulars by capital letters and properties or characters by small letters. Thus the difference between *All S is p* and *All Ss are Ps* corresponds to that in ordinary language between *All men are mortal* and *All men are mortals.*

[2] We may leave it an open question whether the denial of '*Something is p*' is to be interpreted as a new mode of judging, or to be accounted for in terms of an assertion of incompatibility.

The fundamental operation of generalization thus would seem to consist in the transition from the simple judgment *S is p* to the general judgment '*Something is p*'. It remains to inquire how this latter type of judgment is constituted.

The judgment *S is p* has been seen to consist in three terms, T the thinker, S the 'subject' and the character *p* being multiply related by a judging relation. What is characteristic of the transition to *something is p* is the elimination of the reference to a particular *S*. Further, there would appear to be no reason to suppose that any other particular is substituted in its place. Curiously, a judgment about 'something' does not involve anything in particular as its 'subject'.

The absence within the judgment '*Something is p*' of any particular corresponding to 'something' compels us to recognize two fundamentally distinct ways in which we may be said to 'think of' or 'refer to' an object of thought. In asserting that '*This is p*' or that '*Jones is so and so*' the words 'this' and 'Jones' stand for or 'name' some constituent of the judgment. Some individual or thing is thought about in a particularly intimate way. Some element in the fact asserted is also an element in the judgment. In asserting that 'B is a round patch on page 39 of this book' the particular which in fact is a round patch on page 39 of this book is an element in my judgment.

If I assert that '*something* is a round patch on page 39 of this book' I can still be said to be thinking of or referring to the particular B. But I am referring to it in a fundamentally different way. If there had been a dozen round patches on this page I could be said to be referring to each of them even if I had not noticed some of them. But none of them, neither those I had previously noticed nor any of those I had overlooked would have been an element in my judgment.

Especially important is the case in which we are able to assert (i) that there is something which is a round patch on page 39, and (ii) that there is not more than one round patch on page 39. This conjunctive assertion refers to one and to only one thing, but that one thing is, again, not a constituent of the judgment. We may be said to refer to it by description.

In the case cited the reference to a particular by description was subsequent to and derivative from reference by direct acquaintance. But sometimes we refer in thought to something with which we have not previously been acquainted.

Consider the statement: *The first letter in the next chapter of the book you are reading now is an I.* As the author of this statement is the author of the chapter in question the reader will probably be prepared to accept this statement on trust. In accepting it he is holding a belief about something which he has probably not yet observed. If he experiences any doubt he may turn to page 59 and verify the statement. He will then make the judgment *Yes, that is certainly an I.* But in passing from the first judgment to the second he has referred to the subject of his thought in two different ways.

In the first case he was thinking about or referring to the object in question by means of what is expressed by the description—'the first letter in Chapter IV.'. In the second case he was acquainted with it. It was an element in his actual thought. The distinction is of the greatest importance in understanding how we think of things which are not present to mind in that very fundamental way in which the things we actually see and hear are present. By means of descriptions we can think of things in the remote past and the remote future and of things far beyond the reach of our senses. We can think of things in this way even when we cannot in the least imagine 'what they are like'. We have been led to understand that the taste of nectar—the potion of the gods—was extremely pleasant. We can significantly assert that it was pleasant even though we may doubt whether it was at all like honey or anything that we have ever tasted. If, however, we were favoured to dine with the gods we might in the light of actual experience judge 'So *this* is nectar, it is indeed extremely pleasant'. Our merely descriptive judgment would be replaced by a judgment of acquaintance. We should be in the position of a man born blind who received his sight, or of a man born deaf who received the sense of hearing. In such cases merely descriptive references to colour and sound are supplemented by acquaintance with colours and sounds.

The fact that we can think of what is present to the mind seems less mysterious than the fact that we can think of what is not present. Ultimately both are very mysterious facts, but thinking of things by description becomes a little less mysterious on analysis. The point to notice is that we think of what is not present to mind by means of one of its properties symbolized by the descriptive phrase we employ, and this property *is* present to mind.

For simplicity let us consider the case in which the property present to mind is one that really belongs to something which at some time was immediately present to mind but which we have now forgotten. We recall, let us say, a very distinctive aroma which belongs to a flower which we were recently acquainted with, but which we cannot recall to mind. In this case the property is present but the thing to which it belongs is absent. Since, however, the property does belong to something there is a sense in which we are connected, though somewhat indirectly, with the thing to which the character belongs. The connection involves a relation between us and the property and a further relation between the property and the thing it characterizes. For this reason we may describe acquaintance as a more direct relation than reference by description. Acquaintance itself may be an indirect relation, but if so it is less indirect, than reference by description. According as other characters in the judgment vary so we may be said to refer to 'everything that has the property' in question, to 'some or other thing that has the property', or to 'the one thing that has the property'. These will constitute different species of descriptive judgments. It is obvious that no such distinctions can be drawn in the case of acquaintance. There can be neither indeterminateness nor generality of reference in acquaintance. We are always acquainted with a *this*, never with an *any* or an *every*; nor can we be acquainted with anything as a *one and only* of its kind; though, of course, what we are acquainted with may in fact be unique. The notion of reference by description is therefore of fundamental importance in connection with judgments involving the notion of *every*, *any*, or *the* in some of the more important senses of these terms.

We are thus led to notice that the difference between judgments involving acquaintance and judgments involving descriptive reference is commonly marked by differences in the forms in which they are expressed. A judgment involving acquaintance can generally be expressed in the form

This is p,
or *S is p*

(where ' S ' is a proper name). A judgment involving descriptive reference commonly takes one of the forms

The S is p,
or *What is S is p*,

in which S symbolizes the property in virtue of the presence of which to mind the object it characterizes can be referred to. The distinction, it will be noticed, corresponds to the difference between two examples cited of the traditional Singular Judgment. To class together ' This is p ' and ' The S is p ' obscures one of the most fundamental of logical distinctions.

When we employ expressions such as ' this ' or ' that ' demonstratively we are employing them to signify something with which we are acquainted. When, on the other hand, we use expressions such as ' that which is so and so ' or ' the so and so ' or ' a so and so ' we are referring to something by description. This distinction will require further discussion later.

BIBLIOGRAPHICAL NOTES

CHAPTER III.

In this chapter, as in Chapter II., it has been necessary to discuss in a preliminary way a number of difficult and controversial topics. Throughout, a policy of oversimplification has been deliberately adopted ; and the student must be prepared to correct provisional conceptions at a later stage in his studies. (The necessary corrections, however, do not affect the more purely formal aspects of the subject which occupy the next few chapters.) One type of oversimplification is common to two related matters. In the comparison of two facts such as ' This is black ' and ' That is white ' the fundamental consideration is that they differ in two distinctive ways. It is a *convenience* to suppose that this is equivalent to the assertion that such facts are composed of two kinds of ' elements '—particulars and universals. Later, the student must be prepared to inquire whether this latter mode of expression is not seriously misleading. In the same way it has been seen that two judgments may differ in two ways—in 'what is judged' and in the 'mode of judging'. Here, too, it has been a convenience to suppose that this fact may be expressed by saying that a judgment contains two factors—the 'proposition' and the subjective attitudes. This transition also almost certainly needs correction, but a correction which can only be expressed by saying that the proposition is a 'logical construction'. (Cf. Chapter IX., § 4.) For a more technical discussion consult the important symposium between **F. P. Ramsey** and **G. E. Moore, 'Facts and Propositions'** (**Proc. Aristotelian Soc.,** Supplementary Vol. VII.), and the five articles by **John Wisdom** (**Mind,** Vols. XL.-XLII.) on **'Logical Constructions'.**

The *analysis of judgment in terms of multiple relations is due to* **B. A. W. Russell (Philosophical Essays—On the Nature of Truth and Falsehood).** The same author is responsible for the prominence in modern logical theory of the distinction between Acquaintance and Knowledge by Description. The **Problems of Philosophy** (Chapter V.) contains an elementary introduction to this topic—but the files of the Proceedings of the Aristotelian Society, and some of the works referred to in the Bibliographical notes to Chapter IX., contain many related discussions. In the text we have employed the expression 'reference by description' rather than 'Knowledge by description' since the concept applies beyond the field of knowledge proper. It applies equally to the analysis of mistaken beliefs.

Related discussions of all these matters will be found in the general treatises on Logic. Special reference should be made to **Johnson's Logic** (Part I.); **Stebbing, Modern Introduction to Logic,** Chapter IV., **Joseph's Introduction to Logic** and his article in **Mind** (Vols. XXXVI. and XXXVII.): **'What does Mr. W. E. Johnson mean by a Proposition?'**

CHAPTER IV.

COMPOUND PROPOSITIONS.

§ 1. Compound Forms of Propositions.

In the preliminary survey of formal distinctions between propositions it was noticed that compound forms are obtained by combining simpler propositions in various ways. The number of distinct modes of combination is limited. Given two propositions *p* and *q* we shall provisionally distinguish five ways in which they can be connected in a single compound proposition. The modes of connection may be summarized as follows:

(1) The conjunctive mode of combination consisting in joint assertion of *p* and *q*.
(2) The disjunctive mode in which *p* and *q* are asserted to be incompatible.
(3) The alternative mode asserting that either *p* or *q* is true.
(4) The implicative mode asserting that the truth of *p* determines the truth of *q*.
(5) The counter-implicative mode asserting that the truth of *p* depends upon the truth of *q*.

Each of these modes of combination requires explanation.

The Conjunctive Mode of Combination.—The simplest of grammatical conjunctions—the word 'and'—expresses a very fundamental way in which two propositions can be combined. Given two propositions *p* and *q*, we may assert one and abstain from asserting the other; or we may, after due consideration, decide to assert them both: *p* is true ***and*** *q* is true. The two propositions: *The duchess was ugly, The duchess was exceedingly ill-tempered,* may be jointly affirmed in the form: *The duchess was ugly and the duchess was exceedingly ill-tempered* or in the form *The duchess was both ugly and exceedingly ill-tempered.* Propositions with a single subject and two predicates are equivalent to a conjunction of two propositions having the same subject and different predicates. There are other forms of expression which differ importantly in literary qualities and in emphasis but are of the same logical significance, e.g. '*The duchess was not only ugly, she was exceedingly ill-tempered*'. 'But' serves the same logical purpose as 'and'. It is employed when the actual conjunction asserted is contrary to one that might have been expected. Contrast '*He took the money, but he did not realize that it was theft*' with '*He took the money and he realised that it was theft*'.

The use of a special word for expressing logical conjunction is not essential. The mere juxtaposition of statements serves the same purpose. This paragraph may be regarded as a single conjunctive assertion of the various propositions expressed by the sentences individually. The whole book is equivalent to a single proposition jointly asserting all the propositions it contains.

For the purposes of formal logic the conjunction of p with q may be expressed by 'p *and* q' or more simply by the juxtaposition $p\ q$. More convenient, however, is the use of the dot (.) as a symbol of conjunction, so that $p\,.\,q$ stands for the conjoint assertion of the two propositions

p *and* q.

It is self-evident that the conjunction of two propositions is itself a proposition. This self-evidence follows from the nature of a proposition; propositions are among the things which can be conjoined so as to yield something of the same nature as the conjoined elements. The general formula: '*An x conjoined with an x yields an x*' is not always true. It is true of classes. It is true of various other things, but there are things of which it is not true. If we can attach significance to the assertion *A triangle conjoined to a triangle yields a triangle*, the statement is false. It is, however, true of propositions. Hence we must recognize a compound proposition of the conjunctive type.

The Disjunctive Mode of Combination.—'*You can't have it both ways—that a tariff would produce revenue equal in amount to the total tax levied and that the tariff would benefit industry to a like amount.*' Statements of this kind express a disjunction or incompatibility. It is asserted of two given propositions that they cannot both be true.

The disjunctive way of combining two propositions may be expressed in the form:

Not-both p and q,

or by $p|q$.

The latter mode of representation can be read as the former, '*not-both p and q*', or simply '*p stroke q*'. It should be noted that a disjunction is compatible with a denial of both propositions. '*Not-both p and q*' is consistent with '*neither p nor q*'. '*You can't say in the same breath both that he is incompetent and that he is a clever rogue*' is equivalent to the assertion that at least one of these statements is false. They both may be false.

The Alternative Mode of Combination.—Two propositions are combined alternatively when it is asserted that one or the other is true. '*Either he saw a ghost or he has a very vivid imagination*'. We can often be sure that such a compound proposition is true when we are wholly uncertain which of the alternatives is in fact true. We frequently employ the alternative form of a proposition in narrowing down a range of suggested possibilities. The form is simply expressed by the ordinary use of '*Either . . . or . . .*':

Either p or q,

or by $p \vee q$.

The latter mode of expression is read in the same way as the former.

As the disjunctive is not inconsistent with 'neither' so the alternative is not inconsistent with 'both'. In the suggestion '*You must either borrow that book or steal it, it's worth reading*' the alternants are not mutually exclusive as all who lend books know.

This is a matter on which there has been some confusion. As noted previously,[1] the compound forms traditionally recognized were the 'hypothetical' and the 'disjunctive'. The former confused the two forms to be distinguished as *implicative* and *counter-implicative;* the latter confused the disjunctive and the alternative. The so-called disjunctive form combined the features of the disjunctive and those of the alternative. It was in effect, therefore, a conjunctive proposition in which the conjuncts were respectively a true alternative and a true disjunctive. It is expressed by

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[1] Ch. III., p. 49.

$$(p \vee q) . (p|q).$$

One reason for this confusion is no doubt the fact that in everyday usage 'Either . . . or . . .' sometimes expresses this combination of conditions. This, in turn, is due to the fact that from the nature of the case the alternants are also sometimes mutually incompatible.

· '*To be or not to be, that is the question*'. Hamlet's alternatives were not only jointly exhaustive of the possibilities, they were unfortunately mutually exclusive. Similarly, if we know that '*John arrived either by the* 10.50 *train or by the* 11.40' we may presume that he did not come by both. Logically, however, the fact that he arrived by the 11.40 would not provide him with a cast-iron alibi had murder been committed on the 10.50. This depends upon the spatial and other temporal circumstances of the case.[1]

The Implicative Mode of Combination.—Two propositions may be combined in the manner expressed by such statements as '*If Dr. Jones is out you will find his partner Dr. Robinson in*'. This is the implicative mode of combination expressed by

If p then q,

or by

$$p \supset q,$$

which is read '*p implies q*'.

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[1] The confusion of alternative and disjunctive propositions has had other unfortunate consequences. The usage here followed is that of Johnson. Other logicians, notably the mathematical logicians, eliminated the disjunctive element from the alternative form, but preserved the name, 'disjunction' to express what is here meant by 'alternation'. This was an unfortunate choice of terminology, since the alternants, not being mutually exclusive, cannot be said to be disjoined. It had the further consequence that when true disjunction was later introduced by these logicians a new term was required. The disjunctive was then described as the 'stroke function'. Consequently in these works we meet what is naturally to be described as an alternative proposition under the title of 'disjunction', and what is naturally described as a disjunction under the title of 'the stroke function'.

It is to be observed, however, that there is in logic no divergence of doctrine on the point. All logicians agree that $p \vee q$ must be read so that p and q are not mutually exclusive and that $p|q$ is consistent with neither p nor q. *The confusion is merely one of terminology.*

A feature of great importance which distinguishes this relation between two propositions from the relations of conjunction, disjunction and alternation is that here the relation is not symmetrical. There is no difference between the compound proposition $p \,.\, q$ and the proposition $q \,.\, p$, nor between $p|q$ and $q|p$, nor between $p \vee q$ and $q \vee p$. But there is a vital distinction between $p \supset q$ and $q \supset p$. To pass from one to the other is not an infrequent fallacy. It is a fallacy of the same order as that of treating alternatives as mutually exclusive. We cannot infer from the fact *if Dr. Jones is out Dr. Robinson is in* that *if Robinson is in Jones will be out.*

The implicative form of the compound proposition corresponds to the traditional hypothetical proposition.[1]

The Counter-Implicative Mode of Combination.—If the implicative form of combination be defined as one in which a given proposition is united with one dependent, in a certain sense, upon it, then it would seem reasonable to distinguish as a distinct mode of connection that in which the given proposition is combined with one upon which it is dependent. That is the counter-implicative form.[2] There is no distinctive grammatical conjunction for expressing this mode of connection. The fact that p depends upon q is commonly expressed by saying that *q implies p.* For certain purposes, however, it is convenient to express this relation between the two propositions p and q preserving the same order of statement as in the other forms. The conjunction '*since*' approximates to what is required, but '*p since q*' carries the further implication that q (and therefore p) is true. These implications are absent in the purely hypothetical form, *If q then p.* Sometimes the bare fact that p is implied by q is written '*p if q*'; e.g. '*There will be rain if the wind drops*'. We cannot, however, distinguish this from the direct implicative '*If the wind drops there will be rain*' unless the order in which the two components are given is significant. The distinction between the implicative and the counter-implicative forms is only possible when we know which component is the one *primarily given*, or when an equivalent implicative proposition is also given. Thus given any proposition we can construct another of the implicative form by combining it with another which it implies, and we can construct a counter-implicative compound proposition by combining it with one which implies it. Again, given *if a is true then b is true* as an implicative proposition we can say that *if b is false a is false* is a counter-implicative. This will be seen more clearly when we consider the relations between the different compound forms.

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[1] With the reservations noted in Ch. III.

[2] Introduced by Johnson, but not recognized by all logicians.

The fact that there is no commonly employed conjunction available to express this relation of one proposition to another upon which it is dependent makes it desirable to employ a special symbol for this mode of connection. The fact that symbolic logicians have not generally recognized this form compels us to introduce a symbol that is not in common use. Since the relation to be expressed is the converse of that involved in the implicative form we may simply reverse the symbol there employed, writing the counter-implicative

$$p \subset q$$

which can be read '*p if q*' or '*p is implied by q*'.[1]

§ 2. Relations between the Compound Forms.

The compound forms are not wholly unconnected with each other. It is obvious that $p|q$ is incompatible with $p \,.\, q$. It will be found, further, when we take into account the contradictories of the given propositions that certain equivalences[2] hold between compound forms.

Every proposition has a contradictory. The contradictory of a proposition p may be symbolized by *not-p* or by $\bar{p}$ or by $\sim p$. Given two propositions p and q and their contradictories, $\sim p$ and $\sim q$, there are four ways in which they can be conjunctively combined:

$$p \,.\, q, \quad p \,.\, \sim q, \quad \sim p \,.\, q, \quad \sim p \,.\, \sim q.$$

It is then easily seen that the disjunctive combination of p with q, $p|q$ is equivalent to the alternative combination of $\sim p$ with $\sim q$, i.e.

$$p|q \equiv \sim p \vee \sim q.$$

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[1] It is important to note carefully that this symbol is not employed for this purpose outside this book. It occurs in the 'Principia Mathematica' to symbolize an analogous relation between *classes*. No confusion is likely to result. The symbol will not be required after the present chapter except for one passage in Chapter VI.

[2] Two propositions are said to be equivalent ($\equiv$) when each implies the other. Cf. Ch. VI.

If it be the case that *you cannot both have your cake and eat it* then it is also the case that *either you do not have it or you do not eat it.* This is in turn equivalent to the implicative proposition, *If you have your cake then you do not eat it,* which again is equivalent to the counter-implicative proposition that *you do not have your cake if you eat it;* i.e.

$$p \mid q \equiv {\sim} p \vee {\sim} q \equiv p \supset {\sim} q \equiv {\sim} p \subset q.$$

Since no conjunctive proposition either implies or is implied by any of the other forms, and since each of the other four forms are related by mutual implication, we may distinguish the latter as ' the composite forms '.

The relations between the five forms obtained by combining p, q, ${\sim} p$ and ${\sim} q$ in the five possible combinations are shown in the following table:

Implicative.	Counter-Implicative.	Alternative.	Disjunctive	Conjunctive.
$p \supset q \equiv$	${\sim} p \subset {\sim} q \equiv$	${\sim} p \vee q \equiv$	$p \mid {\sim} q$	$p \,.\, {\sim} q$
${\sim} p \supset {\sim} q \equiv$	$p \subset q \equiv$	$p \vee {\sim} q \equiv$	${\sim} p \mid q$	${\sim} p \,.\, q$
${\sim} p \supset q \equiv$	$p \subset {\sim} q \equiv$	$p \vee q \equiv$	${\sim} p \mid {\sim} q$	${\sim} p \,.\, {\sim} q$
$p \supset {\sim} q \equiv$	${\sim} p \subset q \equiv$	${\sim} p \vee {\sim} q \equiv$	$p \mid q$	$p \,.\, q$

This table shows that corresponding to every implicative proposition there is an equivalent counter-implicative, an equivalent alternative and an equivalent disjunctive proposition. Each disjunction is contradicted by a corresponding conjunction. Since any proposition which contradicts a given proposition is also contradictory to every equivalent proposition the conjunctive forms are the contradictories of every form on the same line. It will be found, moreover, that propositions on different lines are independent. Two propositions are independent if either may be true or false without implying either the truth or the falsity of the other.

The equivalence of the various composite forms may be more fully grasped by formulating specific rules for inferring any required equivalent form when one is given.

Given an implicative proposition, $p \supset q$.

This is an assertion to the effect that a certain antecedent implies a certain consequent. The following principles then hold:

(1) The denial of the antedecent is implied by the denial of the consequent.

This yields the counter-implicative ${\sim} p \subset {\sim} q$.

(2) Either the antecedent must be denied or the consequent accepted.

This yields the alternative $\sim p \vee q$.

(3) The acceptance of the antecedent is incompatible with the denial of the consequent.

This yields the disjunctive $p | \sim q$.

The contradictory of an implicative proposition consists in the joint acceptance of the antecedent and the denial of the consequent, $p \,.\, \sim q$.

Similarly, given a disjunctive proposition $p|q$.

We can formulate the following principles:

(1) The acceptance of the first 'disjunct' implies the denial of the second.

(2) The denial of the first is implied by the acceptance of the second.

(3) Either the first disjunct or the second must be *denied*.

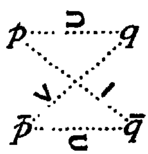
The contradictory of a disjunction is the joint assertion of both disjuncts.

Corresponding principles for determining equivalent forms given a counter-implicative or an alternative proposition can be formulated readily once the relations are clearly grasped.

These principles are not merely of interest to the formal logician. Mistakes in everyday reasoning frequently arise from an imperfect realization of the relations they express. It is important to note, for example, that an implicative proposition is not contradicted by a denial of the antecedent, and that denial of the antecedent does not disprove the consequent. Again, it should be observed that in order to contradict a conjunctive proposition it is not necessary to disprove either of the propositions conjoined. It is sufficient to show that one or other disjunct is false, without asserting which.

In the case of arguments expressed in the form 'p since q' which as we have noted conjoins the counter-implicative proposition $p \subset q$ and q, the contradictory it will be noted is *either* $\sim p \,.\, q$ *or* $\sim q$. It follows that we can 'meet' an argument of this kind in two ways; we can deny q or we can assert that $\sim p$ is conjoined with q. Thus the statement: *War is inevitable since man is instinctively pugnacious*, is met either by denying that man is instinctively pugnacious or by asserting that the abolition of war is consistent with man retaining his instinctively pugnacious nature.

At this point it will be convenient to notice an ambiguity which has been the source of much confusion and controversy. We have seen that the four propositions p, $\bar{p}$, q and $\bar{q}$ are so related that if a certain relation holds between p and q then certain other relations hold between $\bar{p}$ and $\bar{q}$, $\bar{p}$ and q, p and $\bar{q}$. These relations are represented in the following figure.



That is to say, if $p \supset q$, then $\bar{p} \subset \bar{q}$, $\bar{p} \vee q$, and $p|\bar{q}$. Corresponding figures and statements can be provided for the cases $p \subset q$, $p \vee q$ and $p|q$. This, of course, is merely to repeat what is asserted by the table of equivalences on page 63.

Now the expression 'If . . . then' (and consequently the symbol $\supset$) is ambiguous. It is possible to place various interpretations upon it which will involve corresponding variations in the interpretation of $\subset$, $\vee$ and $|$. Two interpretations, in particular, are of fundamental importance.

One of these is that according to which $\supset$ represents what is called 'material implication'. This may be defined as the relation which holds between p and q whenever it is not as a matter of fact the case that p is true and q is false. Suppose, for example, we are in a position to assert, for any reason whatever, *that it is not the case, both that 'This man's name is Smith' is true, and that 'That man's name is Robinson' is false.* This total assertion may be described as a 'matter of fact' disjunction. It can also be stated, *It is not the case both that 'this man's name is Smith' and that 'that man's name is not Robinson'.* This is the matter of fact disjunction $p|\bar{q}$. Corresponding to this, there is a matter of fact alternation, a matter of fact implication, and a matter of fact counter-implication. Material implication is the matter of fact implication 'If this man's name is Smith, then that man's name is Robinson'. It is the relation which holds between p and q when matter of fact disjunction holds between p and $\bar{q}$.

There is, however, another relation much more commonly expressed in ordinary language by the word 'implies', viz. that relation which holds between p and q when *given p alone q* can be inferred. This obviously does not hold in the example cited. Confronted with two strangers, if we are told 'This man's name is Smith' we cannot infer anything about the name of the other man. But now, confronted again with two strangers A and B, we are told that A is the father of B's son. From this fact alone we can deduce something important about B, namely, that B is A's wife. The assertion (p) that A is the father of B's son implies (q) that B is the wife of A. This is clearly a different sense of implies, and is sometimes called strict implication. It is the relation referred to previously as 'entailment'. As with material implication it involves corresponding relation between $\bar{p}$ and $\bar{q}$, $\bar{p}$ and q, p and $\bar{q}$. If p strictly implies q then $\bar{p}$ is strictly implied by $\bar{q}$, we may also say that either $\bar{p}$ or q *must* be the case, and that it is *logically impossible* for both p and $\bar{q}$ to be the case. It is also of importance to notice that in symbolical formulæ in which the implication sign occurs it may sometimes admit of interpretation only in terms of material implication and at other times in terms either of material or of strict implication. For example, it has been said that if p materially implies q then either p is false or q is true. This may be written

$$(p \supset q) \supset (\bar{p} \vee q).$$

Here even where the implication sign within the brackets represents material implication that which occurs between the two bracketed expressions admits of interpretation as representing strict implication. The latter may also be read in terms of material implication, since if $(p \supset q)$ necessarily implies $(\bar{p} \vee q)$ it is also in fact true that it is not in fact the case that the former is true and the latter false. The material implication follows from the strict implication as a species of inference by modal consequence.[1] So, too, if we know that $p \supset q$ and we know that p is true we can infer q, i.e. $\{p \text{ and } (p \supset q)\} \supset q$. The latter implication sign may be interpreted in terms of strict implication, even if the former be interpreted as representing material implication.

§ 3. Compound Propositions and Compound Functions.

The modes of combination dealt with in the preceding section are also described as 'functions'. The conjunctive form $p \,.\, q$ is said to be the conjunctive function of the two propositions p and q; the form $p|q$ is the disjunctive function of the two propositions p and q; the form $p \vee q$ is the alternative function, and so on.

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[1] Cf. Ch. VII.

This use of the notion of function is similar to that employed in mathematics. The familiar and elementary functions of mathematics are functions in which the variables are numbers. In the present logical usage they are propositions. The general conception involved is that of 'functional dependence' or 'concomitant variation'. We say, for example, that the cost of painting a wall is a function of its area. We mean that if we consider a series of walls, 10, 20, 30 . . . square yards in area, we shall find that the cost of painting will vary with the area. Similarly, we might say that a student's proficiency in Logic is a function of the time he devotes to studying it. This means, in the simplest case, that for every variation in time devoted to the subject there will be a variation in proficiency. So, generally, if b is a function of a, variations in a will be matched by variations in b.

But merely to assert that b is a function of a does not tell us how exactly variations in a will be matched by variations in b. We require to formulate some rule in accordance with which we can ascertain what b is when we know what a is. In other words, we wish to be able to assign a specific *value* to b when we are given the specific value for a. In some cases we may find a very simple rule. If the painter agrees to paint at the rate of one shilling per square yard, then for whatever number of square yards we put as the value of a the same number of shillings will give the value of b. But the painter might prefer to base his charges, in part at least, upon the time required for the task. In this case the cost of painting the wall will be a function of the *rate* at which the work is done. The faster a painter works, i.e. the greater the number of yards painted in an hour, the smaller the number of shillings the work will cost. The value of b will vary inversely with the value of a, where a represents rate of work.

This illustrates a further point: A dependent variable may be a function of two or more independent variables. The cost to a contractor who pays his employees on a time rate will be a function of a number of variables such as the rate at which they work and the amount of material they use. If we let c stand for the cost of the work, p for the amount of paint required and r the rate of work of the painter, c will be a function of p

and r jointly. The rule which enables us to ascertain the value to be given to c when we know the values of p and r may be rather complicated, but when discovered it would express *what* function c is of p and r.

Mathematicians are concerned with the various specific relations which may hold between numbers. Some of the simplest functional relations are the following:

$$a = b^2,$$
$$a = \frac{b}{1},$$
$$a = \sqrt{b}.$$

Given two independent variables we obtain functions such as the following:

$$a = b + c,$$
$$a = b \times c,$$
$$a = \frac{b}{c},$$
$$a = b^2 + c^2,$$

and many others of a much more complicated kind.

In any expression such as $a = \sqrt{b}$, the independent variable is called the 'argument' of the function; a, the dependent variable, is called the 'value' of the function. It is, so to speak, the value *par excellence* which we wish to ascertain.

This usage of 'value' must be distinguished from that in which we speak of assigning values to the variables in an equation. We can speak both of assigning a value to the argument and of ascertaining the value of the value of the function. The latter expression employs both senses of 'value'.

When we speak of 'the function itself' as contrasted with the thing which is a function of something else we are referring to the relation in which the argument stands to the value. The function itself is expressed by the symbols, $\times$, $\div$, $\ldots^2$ $+ \ldots^2$ as contrasted with what is expressed by the variables a, b and c in the expressions $a = b \times c$, $a = \frac{b}{c}$, $a = b^2 + c^2$. In other words, the function is symbolized by the 'constants', not by the variables. The symbol for an undetermined function of one argument is

$$f(x).$$

If we require to symbolize several distinctive functions we may employ alternative expressions of the same kind, such as $f(x)$, $g(x)$, $\phi(x)$, where the symbols outside the bracket stand for the undetermined function and the symbol inside the bracket for the undetermined argument. It might be thought that such an indeterminate expression could serve no useful purpose. But it is not wholly indeterminate. It shows the kind of function we are concerned with. In the present example the symbols represent functions of a single argument. The expression $f(x, y)$, on the other hand, would stand for a function involving two arguments, i.e. for functions of the kind $a = x + y$. Similarly, $f(x, y, z)$ would stand for any function of three arguments such as

$$a = \left(x^2 + \frac{y}{z}\right).$$

It may be helpful also to compare the mathematical with the biological conception of function. In the latter there are two implications, one of which is wholly independent of the mathematical notion, the other being identical with that of the mathematical usage. In contrasting function with structure the biologist is concerned with the distinction between the properties of bodily organs which can be expressed in statements of the form '*S is P*' and the properties expressed in statements of the form '*S does something*', i.e. with the difference between the *nature* of the organ and its *activities*. There is, however, the further suggestion that the functioning of an organ is *functionally* dependent upon its structure in the ordinary mathematical sense. Such functions or activities as vision or hearing depend upon the structure of the eye and the ear. Corresponding to variations in the structure of the lens we find such resultant differences as those between near and long sightedness. Differences in the ability to perceive colours are in the same way functionally dependent upon the physical and chemical structure of the retina. When it is suggested that function determines structure what appears to be meant is that the structural properties of the organs are functionally dependent upon its activities, i.e. that present structure is determined by past functions.

The notion of a function is employed in Logic in connection with the theory of propositions in two ways. There are functions of which both the arguments and the values are propositions, and there are functions of which the values but not the arguments are propositions. The former are relevant to the subject of the present chapter; the latter will be discussed in Chapter IX.

Consider a mathematical function involving two arguments,

e.g. $a = b^2 + c^2,$

or $a = b + c,$

or $a = b \times c,$

and let a, b and c stand not for numbers, but for propositions. Can we place an interpretation upon the expression?

Clearly in the first case the result is nonsense. We can attach no meaning to 'the square of a proposition'. In the second and third examples, however, it is possible to attach meanings to $+$ and $\times$ so that these signs can stand for relations between propositions and such that a given proposition can be said to be a function of the two arguments b and c. It will be found that there is an analogy between the addition of numbers and the alternative combination of propositions, and an analogy between the multiplication of numbers and the conjunctive combination of propositions. Consequently, $p \vee q$ can be described as the logical sum of the propositions p and q, and $p \,.\, q$ can be described as the logical product of p and q.

Quite apart from such analogies it is evident that the expressions

$$p = q \vee r,$$
$$p = q|r,$$
$$p = q \,.\, r,$$
$$p = q \supset r,$$

are functional expressions. Thus $p = q \vee r$ may be read 'p is the proposition resulting from the alternative combination of q with r'. Similarly, $p = q|r$ may be read 'p is the proposition resulting from the disjunctive combination of q with r'.

In every case we can say $p = f(q, r)$, meaning p is some function of the two arguments q and r. Corresponding to every value we may choose to substitute for q and r we shall obtain a corresponding proposition p. Let us take the two propositions

Jones is out (for q)
Robinson is out (for r)

and ascertain the proposition p which is determined according to the specific function involved.

The conjunctive function, $p = q \,.\, r$, will determine the proposition: *Both Jones and Robinson are out.*

The disjunctive function, $p = q|r$, will determine the proposition : *Jones and Robinson are not both out.*

The alternative function, $p = q \vee r$, will result in *Either Jones or Robinson is out.*

The implicative function, $p = q \supset r$, will determine *If Jones is out so is Robinson.*

The counter-implicative function, $p = q \subset r$, will determine *Jones is out if Robinson is.*

The five ways of combining propositions into compound propositions may therefore be described as five compound functions involving two arguments.

We may have functions of more or less than two arguments. The negative proposition, *not-p*, $\bar{p}$, or $\sim p$, is said to be the contradictory function of p. It thus illustrates a function of one argument. If we write

$$q = \sim p,$$

q being the value of the function and p the argument, then whatever proposition we substitute for p we shall obtain another proposition which stands in a determinate relation to p, viz. the proposition which contradicts p. If for p we substitute the proposition *All men are mortal*, the proposition q so determined will be one that asserts that *It is not the case that all men are mortal.* If p is replaced by the compound proposition *Either the world is round or the whole of scientific evidence counts for nothing*, then q becomes *It is not the case that the world is round and it is not the case that the whole of scientific evidence counts for nothing.*

The alternative combination of two propositions into the single proposition $p \vee q$ has, as we have noted, a certain analogy to addition so that $p \vee q$ can be described as the logical sum of the two propositions p and q. The logical sum of a proposition and its contradictory $p \vee \sim p$ has the property of exhausting all the possibilities of the case. The contradictory function therefore suggests that there is a logical analogue to subtraction. If there was any independent way of defining ' all the possibilities of the case ' we might define one negative function of p as

$$\text{All possibilities} - p,$$

i.e. as the proposition that remains when p has been denied and thereby ' subtracted ' from all possibilities. Since, however, all possibilities cannot be defined except by reference to the meaning of $p \vee \sim p$ it is necessary to take $\sim p$ as a primitive and undefined concept.

Any function of a set of propositions p, q, r . . . which is such that the truth or falsity of the function is determined solely by the truth or falsity of the propositions of which it is a function is said to be a ' truth function ' of these propositions. The truth or falsity of a proposition or function is spoken of as its ' truth value '. Thus if the truth value of p is determined the truth value of $\sim p$ is likewise determined. If the truth value of p is truth then the truth value of $\sim p$ is falsehood. So, too, given two propositions p and q the truth value of the alternative function $p \vee q$ is truth if p is true or if q is true. Its truth value is falsehood if that of both p and q is falsehood. There is, as we shall see in Chapter IX., an elaborate system of compound propositions the truth values of which are in this way determined by the truth values of the elements in this system. Such a system may be described as a ' truth value system '.

BIBLIOGRAPHICAL NOTES.

CHAPTER IV.

An attempt has been made in this chapter to deal in a neutral way with the special region of fact common to traditional and modern logic. The concepts are in the main those of **Johnson** (**Logic**, Part I, Chapter III.). The symbolism is that of **Whitehead** and **Russell's Principia Mathematica.** Such hybridization is not without its dangers. Sufficient warning has perhaps been given in the text with regard to the diversity of interpretation of which the symbolism admits—a fact which points the moral that technical symbolism is not itself a perfect guarantee against inconsistency and vagueness of thought. The formulation here adopted is not, however, essentially controversial. Many of the divergences in exposition among contemporary writers consist in merely terminological differences or in differences in respect of what is taken as primitive and what is derivative, where alternative selections are equally legitimate.

The concept of a ' function ' has been discussed here in some detail owing to the importance it assumes in modern logical doctrine (cf. Chapters IX. and XVI.). On the general mathematical notion consult **A. N. Whitehead, Introduction to Mathematics** (Home University Library) Chapter XI., and **W. S. Jevons, Principles of Science.**

On the meaning of ' implication ' compare **L. S. Stebbing, Modern Introduction to Logic** (Chapter XII.), **G. E. Moore, Philosophical Studies** (Chapter IX., ' External and Internal Relations '), and **B. A. W. Russell, Introduction to Mathematical Philosophy** (Chapter XIV.). For a full treatment of the subject in relation to ' truth value systems,' and the principles of symbolic logic generally, the student should proceed to a study of the **Symbolic Logic** of **C. I. Lewis** and **C. N. Langford.**

CHAPTER V.

THE TRADITIONAL CATEGORICAL SCHEME.

§ 1. The Four Categorical Forms.

THE most general principles of logic concerning propositions do not depend upon an analysis of the proposition. They are independent of the internal constitution of the propositions considered. Given a set of propositions of any type whatever, other propositions can be constructed in the manner already described. Between the different types of compound propositions which result there are certain interesting relations and inferential possibilities which are not without practical importance. Some of these have been noticed in the preceding chapter. It will be convenient at this point to examine certain differences in internal form.

In a systematic exposition, disregarding all historical precedents, we should naturally commence with really simple propositions, i.e. with those which correspond to ' atomic facts '. Combining these by the fundamental logical conjunctions we obtain the body of ' elementary propositions '. The next stage would be to study ' general ' propositions. In such a systematic exposition traditional logic is contained by implication. It occupies a very inconspicuous position, its terminology is almost entirely abandoned and its conceptions have undergone radical transformation.

On the other hand, there are certain advantages in following the traditional lines of exposition. It is important at least to see what it is that modern logic endeavours to replace—or rather to *generalize*. This, however, is not the only advantage. In the logic of tradition there is much of permanent interest and value. Its doctrines are not in general mistaken, but merely limited in scope.

The traditional exposition has in large measure assumed the distinctive form it possesses on account of the difficulty of determining the appropriate simple propositions from which the study of logic might appropriately begin. A simple proposition has been defined as one which does not contain other propositions as constituents. At first sight, it might appear to be very easy to recognize a simple proposition when it is encountered. We are tempted to say that statements like '*All men are fallible*' obviously express simple propositions, whereas such statements as '*If the soul is immortal then it is independent of the body*' obviously express compound propositions. But in this we have probably made the tacit assumption that simple grammatical sentences (i.e. sentences which do not contain sentences as constituents) always express simple propositions. This, however, is plainly false. If it be said '*Some only of the students satisfied the examiners*' the sentence is grammatically simple, but it expresses the compound proposition: '*Some students satisfied the examiners and some students failed to satisfy the examiners.*'

Hence, although it might seem to be more in accordance with good logical order to start with the study of genuinely simple propositions and proceed to more complicated forms, we shall find it practically easier and more convenient to commence with simple statements and inquire into the constitution of the propositions they express. We may hope to reach propositions which are really simple only as a result of analysis. In traditional logic propositions dealt with as simple have in general been those expressed by simple grammatical statements.

Four forms have been selected for detailed study, largely on the assumption that everything or nearly everything which anyone could wish to assert admits of being expressed in one or other of these forms. If this assumption were correct it would follow that every argument depends for its logical cogency upon the implications of these forms. The four forms are exemplified by the following propositions, against each of which is placed the name by which it is technically described:

All explorers are courageous	Universal affirmative.
No stockbrokers are women	Universal negative.
Some books are edifying	Particular affirmative.
Some truths are not believed	Particular negative.

By the ordinary procedure for eliminating what is logically irrelevant we may obtain appropriate symbolical expression for the logical structure of the four forms :

Universal affirmative	*All S is P*	*SaP*	A
Universal negative	*No S is P*	*SeP*	E
Particular affirmative	*Some S is P*	*SiP*	I
Particular negative	*Some S is not P*	*SoP*	O

The symbolical formulæ of the second column are obtained merely by substituting variables in the place of the words which express the 'matter' of the proposition. The remaining words, *All* . . . *is* . . ., *Some* . . . *is not* . . ., etc., express logical constants. In the alternative symbolism shown in the third column the logical constants also are expressed by symbols, the vowels standing for what is signified in ordinary language by *All* . . . *is* . . ., *No* . . . *is* . . ., *Some* . . . *is* . . ., *Some* . . . *is not* . . . . The final column gives the briefest symbols by means of which we can refer to these propositional forms. The letters A and I are the first and second vowels respectively of the Latin verb 'affirmo' (I affirm), E and O are the corresponding vowels of 'nego' (I deny). Hence, of course, their use also in the symbolism of the third column.

In the constitution of each of these propositional forms five features call for notice: the two terms, 'subject' and 'predicate', the mode of connection of subject and predicate, the 'quantity' and the 'quality' of the proposition.

Every proposition of the traditional categorical form contains two and only two terms, the subject and the predicate, symbolized respectively by S and P.[1] Provisionally we may define the subject as the term the proposition is 'about' in the stricter sense of 'about'. The predicate is the term constituted by the material content of what is said about the subject.

The mode of connection of the subject and the predicate is symbolized in the standard formulation by the word 'is', which is called the 'copula' because it links subject and predicate together. In ordinary discourse the word 'is' may be replaced by some other part of the verb 'to be' or by an inflection of some other verb. The significance of the copula varies according to the category to which the subject and predicate respectively belong. But whatever the category of these terms some mode of connection requires symbolization, and this function is performed by the copula.[1]

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[1] In what follows, for reasons which will become clear later, it is not necessary to adhere to the symbolic convention of representing the subject by a capital letter and the predicate by a small letter. This convention is convenient only when we wish to emphasize one interpretation of these propositional forms.

The distinction between a universal proposition (one about *all* or *none*) and a particular proposition (one about *some*) is said to be one of *quantity*. This is a distinction between two kinds of 'generality' in respect of the subject term. Both are 'general' in contrast with propositions which are about an individual denoted by a proper name or by a 'demonstrative pronoun' such as 'This' or 'That'. The signs of quantity are thus the words '*All*', '*No*' and '*Some*'. *No S is P* is universal since it is equivalent to *All S is other than P*. It should be noted, too, that '*Some*' must be interpreted as equivalent to the more explicit expression '*Some at least*'. If we interpreted '*Some politicians are sincere*' as meaning that 'Some politicians are sincere and some are not' the proposition would immediately be compound. We require the word '*some*' in the sense of '*at least some*' in order to express our knowledge when we have reason to suppose that certain members of the class S have a property, P, but have no knowledge concerning the rest.

Propositions are said to differ in 'quality' according as they are affirmative or negative. The quality of the affirmative propositions A and I is shown by the word 'is' standing alone unmodified by any other word. The quality of the negative proposition, O, is shown by the expression '*is not*', that of the E proposition by the word '*No*'. Uniformity of symbolism might have been secured by writing the E proposition '*All S is not P*' but for the fact that this form of words is ambiguous. It is commonly employed to mean '*Not all S is P*' as in '*All that glitters is not gold*'.

It will be noticed that a single word or symbol often performs several functions. 'No' is both a sign of quantity and a sign of quality. The significance of the word 'is' is extremely complex. Alone or in conjunction with other symbols it expresses the quality of the proposition. It is also the sign that an assertion is being made. Any expression of a judgment contains this word or a verb of equivalent force. Contrasted with moods other than the indicative and with phrases such as ' must be ' and ' may be ' the copula also signifies the modality of the judgment. Moreover, it is a part of the symbol for the property predicated of the subject. In an assertion of the form *S is P*, S may stand alone for the thing spoken about but P standing alone does not always adequately symbolize the property. The property predicated is symbolized by ' *is P* '. This is clearly seen in the case of statements such as ' *That piece of wood is ebony* '. ' *That piece of wood* ' being taken as subject, we cannot say that ' *ebony* ' is the property predicated and that ' *is* ' symbolizes the connection between a thing and its properties. *Ebony* is not a property. Similarly, in ' *that is a sapphire* '. A sapphire is a thing, not a property. The properties are expressed by ' *is ebony* ' and ' *is a sapphire* '. The things referred to have the properties of ' *being ebony* ' or of ' *being a sapphire* '.

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[1] It should be noted that the terms 'subject', 'predicate' and 'copula' and the term 'term' itself are all ambiguous, standing either for constituents of the proposition expressed or for the words or other symbols expressing these constituents. Only the context enables us to determine the sense in which these are employed.

Plurality of functions in a word or symbol is very common. It must be constantly remembered in the attempt to determine the full significance of any word. It is found in artificial symbolism as well as in ordinary language.

Further analysis of the significance of the traditional categorical forms requires an examination of the several ways in which these forms may be ' interpreted '.

§ 2. Interpretations of the Categorical Forms.

We are attempting, let us suppose, to determine the precise force of the dictum of Keats, ' *Truth is Beauty and Beauty Truth* '. This is a compound proposition and we may restrict attention to the first part. Let us suppose further that anything which is beautiful is *ipso facto* ' a work of art '. Ruling out the obviously absurd supposition that the statement is to the effect that ' truth ' and ' beauty ' mean precisely the same property, four ways of stating the doctrine call for consideration :

Every truth is beautiful.
Every truth is a work of art.
Where truth is there is beauty.
Where truth is there is a work of art.

It might be thought that these four sentences express precisely the same universal affirmative proposition, but there are certain subtle differences involved which are of importance to the logician. These differences may be described by saying that in the first statement the original assertion receives a *predicative* interpretation, in the second a *purely extensional* interpretation, in the third a *purely intensional*, and in the fourth an *indicative* interpretation. These terms require explanation.

The Predicative Interpretation.—The conception of a 'class' may be provisionally accepted as intelligible without definition. The notion of a class of things such as the class of animals, books or aristocrats, etc., is a familiar one. This notion is essential to the predicative interpretation of the traditional categorical forms. On this reading the subject term of the statement under consideration is taken to stand for the *class* of truths, i.e. of the things that may be said to be true. The predicate term, on the other hand, stands for the *property* expressed by the adjective 'beautiful'. The proposition as a whole consists in the assertion that every member of the class of truths is characterized by beauty. The copula thus stands (*inter alia*) for the mode of connection between a thing and its properties.

We may notice in passing that although the subject term is said to stand for a class the proposition does not directly assert anything about this class as such.[1] It is not the class that is beautiful but each member of the class. There is clearly a distinction between assertions about a class and assertions about its members. If we say that *Anglican bishops have wives* we are not saying that the class has wives. This would make Anglican bishops a polygamous class. But a class so far from being this is not the sort of thing that can have a wife at all. Conversely, when we say that '*bishops can be divided into ascetics and epicureans*' we do not mean that any individual bishop can be so divided but that the class of bishops can. The statement that white sheep eat more than black sheep (because there are more of them) is a jest based on the same distinction. Stated technically the difference is one between reference to a class 'distributively' and reference to it 'collectively'. The distinction also applies to the use of the word 'All'. The statement, '*All the books in the world won't solve your problem even though they all contribute to the solution*', illustrates both the collective and the distributive uses of this word.

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[1] Hence in the case of this reading at least we must modify the provisional definition of the subject as that which the assertion is about.

The Extensional Interpretation.—The distinctive feature of the 'extensional' reading of the categorical forms consists in the fact that both subject-term and predicate-term stand for classes. This necessitates a change in the interpretation of the 'copula' since we cannot assert that one class is the property of another or that the members of one class are properties of the members of another. In the extensional reading we must interpret the copula (in the A form) as standing for the relation of inclusion. '*All truths are works of art*' asserts that the class of truths is contained in the class of works of art. This is a proposition of the relational form $A\,r\,B$: it asserts that two entities are related in a certain way. The negative form may be interpreted as asserting that one class is excluded from the other. The signs of quantity indicate whether the inclusion or the exclusion is complete or partial. 'Partial inclusion' must not be interpreted as implying partial exclusion. '*Some S is P*' asserts that S is at least partially included in P.

The Intensional Interpretation.—In the statement 'Where truth is there is beauty' the word 'truth' is an abstract noun, not the name of anything concrete in the sense in which '*a* truth'—a member of the class of things that are true—is concrete. Abstract names are names of properties considered 'in abstraction from' the things which possess these properties and in abstraction from the classes of such things.[1] On the purely intensional reading of the categorical forms an assertion is made with reference to a connection between two properties.

The words 'where' and 'there' are suggestive of spatial connection, but they admit of a more generalized interpretation. A property may be said to be 'where' anything it characterizes is, though it is 'there' in a different sense from that in which the thing is. It follows that when two properties characterize the same thing one may be said to be where the other is. This concomitance holds even if the thing characterized cannot be said to be in any place at all. *Prima facie*, at least, this holds of a proposition, i.e. of what is said to be *true*.

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[1] No distinction can be drawn between the meaning of 'true' and 'truth' or between the meaning of 'beautiful' and 'beauty'. The difference between the adjective and the abstract noun merely marks two different ways in which the property enters into the proposition.

On the intensional reading the copula stands for the relation between two properties of the same thing, as contrasted with the relation between a thing and its properties and with the relations of inclusion and exclusion between classes.

The constitution of the intensional proposition entails a difference in the significance of the sign of quantity. We cannot distinguish between ' all ' and ' some ' of the same property —all or some truth, all or some beauty. Distinctions of quantity in this case refer to what we might call the ' instances ' or ' occurrences ' of the property. A property ' occurs ' whenever it characterizes or has application to something. We might restate the proposition in the form : *In all cases of the occurrence of truth there is also the occurrence of beauty.*

Alternatively we might express it in the form ' *If anything is true then it is beautiful* '. This use of ' if . . . ' then . . . is not quite that of the compound proposition since the expression ' anything is true ' as here employed is not a proposition. The force of the statement can be brought out by borrowing a symbolic device from the mathematician. We can write ' *If x is true then x is beautiful* '. We can then say that the formula holds ' for all values of *x* ' or that it holds ' for some values of *x* '. This distinction corresponds with the distinction between the universal and the particular forms of the proposition.

The Indicative Interpretation.—As contrasted with the purely extensional and the purely intensional readings the predicative reading is mixed—the subject stands for a class, the predicate for a property. In this respect the indicative reading is similar to the predicative, but in the latter case the subject term stands for a property, and the predicate term for a class. The assertion, ' *Where truth is there is a work of art* ', is equivalent to ' *Wherever the property of truth occurs there is a member of the class of works of art* '. The reading may be called indicative because the occurrence of the property expressed by the subject term indicates the existence of an object belonging to a certain class. This reading does not raise any important new issues. The copula ' is ' again stands for a relation between a property and a thing. Distinctions of quantity in relation to the subject must be interpreted as in the purely intensional reading.

§ 8. Intension and Extension.

The possibility of four interpretations of the categorical forms arises from the fact that general names—which may occur either in the subject or in the predicate of a statement—have a double mode of significance. They are significant, in the first place, inasmuch as they 'connote' some property or set of properties. To understand a general name is to be aware of the properties which anything must possess if this name is to be correctly applied to it. Reference to properties is essential to the notion of 'intension'. The intension of a general name is constituted by a set of properties or attributes.

A general name is also significant inasmuch as it is applicable to some *thing* or set of *things*. It is applicable to anything which possesses the properties comprised within the intension. The set or class of things to which it is applicable constitutes the 'extension' of the name.

There has been considerable controversy as to which attributes are to constitute the intension of a name. Three possibilities may be distinguished:

(i) There is the set of properties which any speaker or writer may suppose the object to possess on any occasion of his use of the name. This is called the 'subjective' intension of the name. It is clear that subjective intension will be very variable. It differs from one speaker to another, and from one occasion to another. It will be relative to a very complex context.

(ii) There is the set of properties to which the *definition* of the term is restricted, i.e. those which have come to be accepted as constituting the meaning of the name by those who use and understand a language correctly. This is the set of properties which the compilers of dictionaries endeavour to ascertain. This is called the 'conventional' intension.

(iii) There is the complete set of properties which are in fact possessed in common by the members of the class of objects to which the name in question may be correctly applied. This has been called the 'comprehension' of the name, or its 'objective intension'. There is no reason why the set of properties comprised within the objective intension should not be infinite in number. It will very commonly include properties which have not been discovered.

The most fundamental of these notions is that of subjective intension, since the conventional intension must be derived in some way from the significance of the term as used by certain speakers on certain occasions. But for most logical purposes it is sufficient to start from the conventional intension. This is also spoken of as the connotation of the term as contrasted with the denotation. The latter is the extension as determined by the connotation. In most cases in which the intension is referred to without qualification it is the connotation that is meant. Connotation determines denotation in the sense that we can only decide whether a thing belongs to a class by ascertaining whether it possesses the properties connoted by the class name.

General names, as possessing both connotation and denotation, are to be contrasted with ' proper names ' which are devoid of connotation. They are also distinguished from proper names in being applicable, *in principle*, to an indefinite number of things. Though applicable ' in principle ' to many things they may, from the nature of the case, in fact be applicable to only one thing or even to nothing at all. Thus ' even number ' is applicable to an indefinite number of things, ' even prime number ' is applicable to only one thing, ' even prime number greater than two ' is applicable to nothing. The general name ' god ' is applicable to many beings, to one being only, or to nothing whatever, according as polytheism, monotheism or atheism contains the truth about the matter. A proper name is functionally incapable of application to more than one thing; and it cannot apply to nothing.

In virtue of this difference general names are employed in conjunction with symbols which have been called ' applicatives ' since they indicate the nature of the application of the general name asserted in the proposition. The ' applicatives ' are the ' signs of quantity ', ' all ' and ' some ' and their synonyms, the articles ' the ' and ' a ', and various other words such as demonstrative adjectives, and pronouns such as ' my ', ' his ', ' your ', etc.

A general name conjoined with the indefinite article constitutes an ' indefinite description '. This is illustrated in the assertion ' *A man called to-day* '. Any phrase of the form ' *A so and so* '—in this usage—constitutes an indefinite description.

In conjunction with the definite article the general name constitutes ' a definite description '. This occurs in statements of the form *The man who called left a message.* Any expression of the form ' *The so and so* ' in this usage constitutes a definite description.

Definite descriptions are classed in traditional logic with proper names as ' Singular Names ' on the ground that their function is to secure uniqueness of reference, and general and singular names are contrasted with one another as mutually exclusive and co-ordinate species of names. This, however, is misleading. The distinction between a general name and a proper name is more fundamental than that between a general name and a uniquely descriptive name. Moreover, general and descriptive singular names are not co-ordinate species. A descriptive singular name is a general name prefixed by an article. The relation between the various conceptions involved may be summarized as follows :

Proper names (devoid of connotation, and not capable of use in conjunction with applicatives).

General Names (connotative, and capable of use in conjunction with applicatives).

(*a*) Used with indefinite article (' indefinite descriptions ').

(*b*) Used with definite article (' definite descriptions ' or ' singular (descriptive) names ').

Proper names it has been said are devoid of connotation. This statement must be qualified by saying that proper names in their primary usage are devoid of connotation. There is, however, an extremely common secondary or derivative usage in which they are equivalent to descriptions. In the assertion ' All " Johns " are men ' what is grammatically a proper name is used with an applicative and is equivalent to the description ' person called John '. Similarly, if we say ' Shakespeare wrote Hamlet ' the statement might be equivalent to ' The man who was named Shakespeare wrote Hamlet ' in which the individual referred to is indicated by a descriptive phrase.

The necessity of interpreting what is *prima facie* a proper name in these statements as being really elliptically expressed descriptive phrases arises from the fact that we can only refer to objects not present to mind by means of their characteristics which are connoted by descriptive phrases. Hence we may define the primary usage of a proper name as that in which it is employed as a symbol for something with which we are acquainted. On this ground it is said that demonstrative symbols such as ' this ' and ' that ' are logically proper names. It is clear that when we refer to something as ' This ' we can give it a proper name in the more usual sense. Suppose that two persons agree that they are seeing the same thing but cannot agree about *what* it is, a ghost, a shadow, or what not. They might say ' Well, let us call it X ' and they can proceed to refer to it by this name without implying that it has any properties whatever. They are then using X not as a variable but as a proper name.

It is often difficult to decide whether a proper name occurs in its primary or in its secondary usage. We must here attend carefully to the judgment expressed by the words and inquire whether the name stands for something with which the speaker is acquainted. The secondary usage presupposes the primary usage in so far as the conception of ' naming ' enters into the description employed. In the explicit expression ' Persons *called* John ', ' The man *named* Shakespeare ' certain individuals are referred to by a description which implies that some one on some occasion ' called them by their name,' i.e used the name as a symbol for something with which they were acquainted. Thus it arises that there is a tendency in contemporary Logic to use the term ' name ' without qualification to stand for the fundamental relation between a symbol and an object of acquaintance. On this usage ' general names ' are not *names* of objects at all, but merely constituents of descriptive phrases.

The use of a proper name enables two persons to refer to the same thing only if each is acquainted with the thing named and if each has associated the name with the thing in question. This agreement is commonly secured by the procedure of ' introduction '. Introduction is effected by the use of such demonstrative adjectives as ' this ' and ' that ' which are, so to speak, temporary proper names restricted to a given context. Their use in communication, moreover, depends upon some other device than language alone. Commonly they are accompanied by a gesture or glance of the eye which directs attention to the object referred to. If telepathy between disembodied minds were a possible means of communication, whatever part language might play in it, it would not involve the use of these demonstrative words. Identification could only be secured by means of descriptive phrases.

There are various ways in which uniqueness of reference can be secured by the use of descriptions. Clearly the use of the definite article by itself is not sufficient: The question, '*Have you read the book of the biologist?*' invites the counter-questions '*Which biologist?*' and '*Which book?*'. But if it be asked: '*Have you read the famous book of the distinguished English exponent of evolutionary biology*', 'Charles Darwin' and the 'Origin of Species' come to mind. There would seem to be several ways in which such identification by descriptions can be brought about.

One way already noticed by implication above is that in which a proper name is an element in the description. Since proper names are employed for a single individual the property of bearing that name commonly provides an exclusive description by means of which a thing can be referred to—an 'exclusive description' being one that applies to only one thing. This method breaks down when a proper name is employed (as Christian names usually are) in different contexts for different persons or things. But it is usually easy to find some property which belongs only to the thing to which reference is to be made. In the case of things which exist in space and time we can refer to the property of being in a certain place at a certain time as when we say '*The man now entering the room is a spy*'. As in the case of demonstrative phrases the significance of such descriptions depends upon the context. 'The room' in this example is equivalent to 'the room in which we now are'. Implicit reference to the present time, the present place or the present speaker is extremely common, if not absolutely invariable.

When no single or simple property provides an exclusive description it is often possible to secure uniqueness of reference by an accumulation of descriptions, as in the parlour game, 'Animal, vegetable or mineral', or in the solution of cross-word puzzles. The object of which the poser of the problem is thinking is one which is 'identified' when the number of clues presented is sufficient to provide a complex description applicable to only one thing within the range of knowledge of the person trying to solve the problem. The interest of a detective story consists similarly in the reader's desire to apply the description 'the murderer' to some one introduced earlier in the story by a 'proper' name.

§ 4. Propositions of Subject-Predicate Form.

The question whether all propositions are 'of the subject-predicate form' is extremely ambiguous. We must therefore examine some of its more important meanings.

(i) In the classification of propositions according to the category of the things to which they refer we have recognized a type in which what is asserted by the proposition is that a substantive is characterized by an attribute or property. It has been noticed, moreover, that on one interpretation of the traditional categorical forms these can be regarded as consisting in a generalized assertion of this kind, viz. as an assertion that some or every member of a certain class possesses or lacks a certain property. The doctrine that every proposition is of this form can be defined by reference to these conceptions. Two points, however, call for notice in connection with the first form of the doctrine under consideration. In the first place it would scarcely be contended that every proposition asserts that a particular individual is characterized by a simple specific attribute. Generalization would be admitted. The second point is that, on the view under consideration, the attributes are of a certain type. They are of the kind that we have described as qualities —properties which do not admit of analysis into any components which themselves are not properties. The view may therefore be stated in this form: Every proposition asserts or denies that an individual thing is characterized by some quality, or it asserts or denies that some or all of the members of a given class is so characterized.

Since we have been compelled to recognize categories other than the category of substantive and the category of quality we must reject this view. Some propositions assert that two or more things are related in a certain way. This would seem to be so obvious that it is difficult to understand why relational propositions should have been overlooked. The mistake is explained by the intimate connection between a relation and a relational property. This distinction, however, may be best discussed in connection with a second form which this general doctrine may assume.

(ii) The assertion *John is the brother of Henry* may be regarded in two ways. It may be taken as an assertion that John and Henry are related by the relation of brotherhood or as an assertion that John has the property of 'being the brother of Henry'. Generally stated, a relational proposition of the form $A\ r\ B$ is equivalent to a proposition of the form $A(rB)$ where the complex bracketed symbol corresponds to a symbol for a property, so that we might equally express the proposition in the form A is P. Properties connected with relations in this way are called 'relational properties'. Their distinctive feature is that they admit of analysis into factors one of which is a relation. Relational properties are one species of what might be called 'complex properties', i.e. properties which admit of analysis into factors some of which belong to a category other than the category of 'quality'. It is not always easy to know whether a property is a quality or a complex property. At first sight 'A is tall' might seem to predicate a simple quality, but it is easily seen the adjective 'tall' must be an elliptical expression of some relational property such as 'taller than the average'. In the case of the properties, 'good' and 'beautiful', there is considerable controversy as to the question whether these are simple or complex.[1]

The fact that relational propositions are equivalent to propositions which assert that something has a property explains in part how it may be supposed that all propositions are of the subject-predicate form. This doctrine has not always been based on the confusion of the two types of properties we have distinguished. It has sometimes been held that relational properties can be analysed into non-relational properties; and it is sometimes held that, though both types of propositions occur, the subject-predicate form is in some sense more fundamental.

We may state the second form of the doctrine that all propositions are of the subject-predicate form as follows: Every proposition either is, or is equivalent to, a proposition asserting that something has or lacks a property (simple or complex); and this form is 'fundamental'. Again we may presume that general propositions of this kind are admitted. The difficulty in this view is the difficulty of seeing in what sense relational properties can be said to be more fundamental than the relations into which they appear to admit so obviously of being analysed.

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[1] We must distinguish complex properties from 'compound' qualities. The latter term may be used for qualities that are analysable, but analysable only into simpler qualities. They are, so to speak, homogeneous in respect of category.

(iii) A third form of the doctrine that every proposition involves the distinction of subject and predicate may be briefly discussed. The distinction here is based upon the way two components in the proposition occur to mind in the course of thought considered as a process in which a judgment comes to be passed.

We have seen [1] that a process of thought normally commences with a question, and that so long as conditions are favourable thought continues until it issues in a judgment which provides an answer to the question raised. Ignoring intermediate stages the sequence is as follows:

Q. Who (or what) broke the vase?
A. The wind broke the vase.
Q. What is that?
A. It was an owl.

Symbolically, Q. Who (or what) is A.
A. B is A.

Or in a slightly more convenient form,

Q. A is x?
A. A is B.

The distinctive feature of a question is that something is given to thought the nature of which is incompletely apprehended. There is something that requires to be ascertained. In the answer this is more fully determined. That from which we start and concerning which something is determined would be said to be the subject of our thoughts. It is common to question and answer. That which is discovered, which occurs only in the answer, may be described as a predicate. It is not improbable therefore that the distinction between the subject of thought and what is ascertained as the result of thought has influenced the interpretation of the distinction of subject and predicate in the proposition reached. The antithesis between what is given for determination by thought and that which determines what is given must, however, be distinguished clearly from subject and predicate in their most usual senses. It has been expressed as a distinction between a 'determinandum' and a 'determinans'.[2] In the application of this distinction to particular cases it will be found that any given proposition will require to be analysed differently according to the question to which it is presumed to supply an answer. Thus in the bare proposition *John is the brother of Henry*, John, Henry or the relation of brotherhood will function as 'determinans' according as the question presupposed be *Who is Henry's brother*, *Who is John's brother* or *How are John and Henry related*. Correspondingly, the subject or 'determinandum' will be 'The brother of John' and 'The brother of Henry' or the pair of men, 'John and Henry'. The last case serves to illustrate a second way in which relational propositions are in a sense reducible to the subject-predicate form—i.e. in a sense in which these terms are identified with 'determinandum' and 'determinans'. We may consider a set of things as a composite subject, and the relation in which they are asserted to stand to one another as the 'predicate'. This view, however, is very different from any given of the traditional doctrine that propositions are always of the subject-predicate form. Moreover, it does not affect the ultimate distinction between subject-predicate propositions in the strict sense and relational propositions. Two fundamentally different types of predicate remain.

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[1] Cf. Ch. II., p. 30.

[2] Johnson's 'Logic', Pt. I., Ch. I., § 5. Johnson appears to define 'subject' and 'predicate' in terms of this distinction.

The question as to the universality of the distinction is primarily a matter of psychological analysis which has no bearing upon the formal treatment of inference.

In distinguishing between four interpretations of the categorical forms it is natural to inquire whether one interpretation is of greater logical importance or more fundamental than the others. One interpretation may be more important than another in various ways and for various reasons, but when it is said to be more fundamental a special kind of importance is generally intended. One interpretation is more fundamental than another if it is of greater logical simplicity, i.e. if it requires fewer indefinable concepts. The less fundamental require the introduction of new concepts.

The claim that the predicative interpretation is the most fundamental or important is derived in part from the fact that categorical propositions in this form are obtained by generalization from propositions of the form S is P, where S stands for a uniquely determined individual thing and P for a simple quality.

It is presumed that no further analysis can be given of the experience in which we apprehend that an object with which we are acquainted possesses a certain quality. It is certainly the case that under these circumstances we do not directly cognize the fact that the object in question is a member of a class, the class of all objects possessing the quality in question. But this explanation itself exhibits the fact that the ordinary categorical forms are not absolutely elementary. The process of generalization is at least required, and some account must be given of the transition from propositions about individuals to propositions concerning a class to which the individuals belong.

A second and distinct respect in which importance is claimed for the predicative form is that it corresponds with the 'natural' or normal form in which judgments actually occur--it being tacitly presumed that other forms are in some way artificial. The actual frequency of occurrence of any given form is, however, not of great logical importance; and the suggestion that other forms are 'unnatural' must be rejected. It is not possible for the logician to invent logical forms. He is in every case concerned with the forms actually discoverable by the analysis of thought.

Logical simplicity pertains to the purely extensional form inasmuch as subject and predicate receive a uniform interpretation. The form cannot, however, be regarded as not admitting of further analysis. The notion of a class is in some way derivative. The form is of importance in that it provides a basis for a relatively simple formal treatment. Class relations are easily apprehended and admit of diagrammatic representation. The general theory of classes provides an important branch of logical theory, which in turn provides the basis for a department of the theory of statistics of considerable scientific value.

The purely intensional forms provide a nearer approximation to an analysis favoured by many contemporary logicians. We have noticed that on this interpretation the A form may be expressed in the form 'if anything is S then it is P' or 'if x is S then x is P'. An expression of the form 'x is a so and so' has been called a 'propositional function' and, on the view here referred to concerning the analysis of the general proposition, what is expressed by the propositional function is taken as the ultimate concept in terms of which the forms are to be defined. This view will be considered later.

The indicative reading has no claim to special importance. It requires notice as one of the logical possibilities which results from the two modes of significance of general names.

In the detailed study of the special doctrines of traditional formal logic it will be convenient to adopt in the main the purely extensional reading. Three questions arise for discussion: (1) How are the four propositional forms related in respect of mutual consistency? (2) What inferences may be drawn from each form singly? (3) What inferences are possible when two or more propositions are taken in conjunction? The answer to the first of these questions is contained in the traditional doctrine of 'Opposition', that to the second question in the doctrine of 'Immediate Inference', and the answer to the third in the doctrine of the 'Syllogism'. These topics will occupy the three succeeding chapters.

BIBLIOGRAPHICAL NOTES.

CHAPTER V.

Chapters V. to VIII. are concerned with what is now commonly designated (sometimes with a suggestion of disparagement) 'traditional' or Aristotelian Logic. The present account, subordinating historical and philosophical to the more purely formal considerations, might more accurately be described as concerned with the formal logic of the traditional categorical scheme of propositions—the logic, that is, of the four forms A, E, I and O.

The exposition closely follows that of **J. N. Keynes, Formal Logic.** The only excuse for attempting to say again what has in that work been so well and finally said lies in the possibility of some condensation for the purposes of a brief general survey, and the possibility of some restatement where alternative accounts are possible.

For the Aristotelian Logic proper the student should read firstly Aristotle himself and then **Joseph, Introduction to Logic. W. E. Johnson, Logic,** Parts I. and II., provides the most authoritative account which both maintains essential continuity with tradition and incorporates distinctively modern features. The relations between traditional and modern logic are discussed in **Stebbing, Modern Introduction to Logic,** and **R. M. Eaton's General Logic.**

CHAPTER VI.

OPPOSITION.

§ 1. Consistency and Opposition.

THE thoughts that pass through our minds may be coherent or incoherent. They may hang together by logical connections or they may lack this coherence. They may on the other hand be mutually incompatible.

The same is true of a system of beliefs. We may hold a number of beliefs which hang together in the sense that if one is disproved others will be affected. It is possible also to hold a set of beliefs which are disconnected, so that one may stand or fall without disturbing the rest. Again, it is possible to hold two beliefs that are incompatible without observing their inconsistency. How this is possible is a curious psychological problem, but the fact would seem to be indubitable.

Leaving aside the psychological aspect of this subject we may inquire more closely into the nature of logical coherence and incoherence, consistency and inconsistency between propositions. These relations are more complex than the simple dichotomous divisions would suggest.

Given two propositions there are seven possible ways in which they may be related in respect of logical consistency. These relations range from complete mutual dependence, through independence to absolute incompatibility. The following are the seven relations :

I. *Co-implication or Equivalence.* Two propositions are said to be co-implicant, equivalent or equipollent, when they are so related that if either be true the other must be true also, and if either be false the other will necessarily be false also.

II. *Super-implication or Super-alternation.* If, of two propositions, the truth of the first implies the truth of the second (but not *vice versa*), the former is said to be super-implicant or super-altern to the second.

III. *Sub-implication or Sub-alternation.* If, of two propositions, the truth of the first is implied by but does not imply the truth of the second, the former is said to be sub-implicant or sub-altern to the second. This is the converse of the relation of super-implication. If p is super-implicant to q then q is sub-implicant to p.

IV. *Independence.* Two propositions are said to be independent when neither the truth nor the falsity of one implies the truth or falsity of the other, i.e. either may be true or false without affecting the truth or falsity of the other.

V. *Sub-opponency or Sub-contrariety.* Two propositions are said to be sub-opponents or sub-contraries if one or other must be true and both may be true, i.e. they cannot both be false but are consistent.

VI. *Super-opponency or Contrariety.* Two propositions are said to be super-opponents or contraries if they cannot both be true but both may be false.

VII. *Co-opponency or Contradiction.* Two propositions are said to be co-opponents or contradictories when they cannot both be true and they cannot both be false, i.e. if either be true the other must be false, and *vice versa.*

The second titles are the traditional names for these relations—the first titles were introduced by Johnson to draw attention to the symmetrical way in which the series is distributed around the central relation of independence.

It will be noticed that if any of the first five relations hold between two propositions they may be said to be consistent; if either of the last two hold the propositions are inconsistent. Again, if any of the first three relations hold the propositions are coherent, if any of the last four they are incoherent.[1] Independence combines consistency with the absence of coherence.

Technically the relations are described as '*relations of opposition*'. They are all of common occurrence in everyday discourse. Consider, for example, the following situation:

Six speakers are engaged in argument. The dispute concerns the explanation of the fact, presumed to be admitted by all, that the social organizations of men in clubs, trade unions, etc., have been much more successful than the corresponding organizations of women. The argument is precipitated by Algernon who remarks, '*The fact is that men, unlike women, are essentially co-operative*'.

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[1] Incoherent or at least non-coherent. The word is ambiguous, sometimes carrying the implication of incompatibility.

'*Not at all*', retorts Ernest, an uncompromising feminist, '*men are selfish and essentially unco-operative. Their organizations are based upon individual self-interest*'.

'*I disagree*', says Charles, '*there is not a man in the world who does not co-operate in some ways with his fellows*'.

Oliver interposes: '*Surely you admit that at least some men are not co-operative?*'

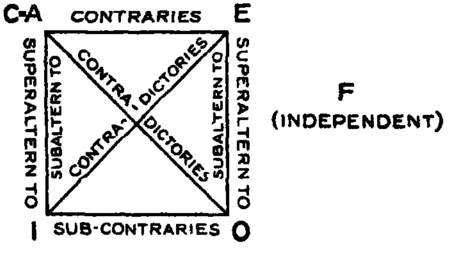
'*But most are*', asserts Ian.

'*The truth is that men are much more intelligent than women*'. This comes from Frederick.

Examining the relations between these assertions we find that the statements of Algernon and Ernest are 'super-opponent' or 'contraries'. Certainly both cannot be right, but they may both be wrong. If some men are co-operative and some men are not the two extreme generalizations will both be false.

The assertions of Algernon and Oliver are mutually contradictory. They cannot both be true and they cannot both be false. The same relation holds between the assertions of Ernest and Ian. The statement of Ian is sub-altern to that of Algernon. Conversely, Algernon's statement is super-altern to that of Ian. If Algernon is right, then Ian must be, but if Ian is right, it does not follow that Algernon is. Similarly, Oliver's statement is sub-altern to Ernest's, and Ernest's statement is super-altern to Oliver's. The statement made by Charles may be taken to be equivalent to Algernon's. Each implies the truth of the other.

The distinctive feature of the contribution of Frederick to the discussion is that it is consistent with the truth, and with the falsity of any statement made by the other speakers. It is irrelevant to the point at issue, and therefore illustrates the relation of logical independence.



We might represent the relations of opposition between the speakers in the accompanying diagram. Frederick, who for all logical purposes, was engaged in soliloquy, is represented on the right. Any other position of independence would be equally appropriate.

Ignoring the statements of Charles and Frederick we may next reduce the four remaining statements to the traditional categorical forms:

Algernon asserts: *All men are co-operative*, i.e. a proposition of the form A.

Ernest asserts: *No men are co-operative*, i.e. a proposition of the form E.

Ian asserts: *Some men are co-operative*, i.e. a proposition of the form I.

Oliver asserts: *Some men are not co-operative*, i.e. a proposition of the form O.[1]

Interpreting the diagram now as representing the relations between the four propositional forms A, E, I and O we obtain the traditional Square of Opposition first constructed by Aristotle and reproduced in almost every subsequent work on Logic. This square summarizes the following facts:

No two of the traditional forms are independent and no two equivalent.

The two universal forms are super-opponent.

The two particular forms are sub-opponent.

The universal form is super-implicant to the particular of similar quality, the latter being sub-implicant to the former.

Universals and particulars of contrasted quality are co-opponent.

§ 2. Four Interpretations of the Square of Opposition.

The relations of opposition have been defined in some cases in terms of implication, in others in terms of alternation, and in others again in terms of disjunction. Thus sub-alternation was defined in terms of implication, contraries and contradictories in terms of alternation and disjunction.

Since, however, any proposition stating an implication is equivalent to one stating a counter-implication, to one asserting an alternation and to a disjunction, it follows that we could define each of the relations of opposition consistently in terms of one of these conceptions alone. Four equivalent definitions could thus be obtained. This procedure is not commonly adopted since, for psychological reasons, one of these relations will be the more obvious in one connection and another relation more easily seen in a different connection. The implicative relations are perhaps slightly more obvious in sub-alternation, the disjunctive more obvious in contradiction.

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[1] These propositions are not strictly equivalent to the given assertions. In the process of reduction distinctions of modality and other important characters are sacrificed. What remains, however, is part at least of what is asserted by the four speakers.

In expository definitions it is natural to select from all the logically equivalent definitions that which is most easily grasped. If logical simplicity and symmetry were all-important considerations we could define each of the relations in terms of the same fundamental notion. The equivalence of the different definitions may be illustrated by considering the four possible definitions of Contradiction:

p is the contradictory of q
$\equiv$ If p then $\bar{q}$ and If $\bar{p}$ then q i.e. $(p \supset \bar{q}) . (\bar{p} \supset q)$.
$\equiv$ If q then $\bar{p}$ and If $\bar{q}$ then p i.e. $(\bar{p} \subset q) . (p \subset \bar{q})$.
$\equiv$ Either $\bar{p}$ or $\bar{q}$ and Either p or q i.e. $(\bar{p} \vee \bar{q}) . (p \vee q)$.
$\equiv$ Not both p and q and Not both $\bar{p}$ and $\bar{q}$ i.e. $(p|q) . (\bar{p}|\bar{q})$.

The first definition can also be stated:
If the first is true the second is false.
If the first is false the second is true.

The second definition:
If the second is true the first is false.
If the second is false the first is true.

(These two together are combined in saying
If either is true the other is false.
If either is false the other is true.)

The third definition:
Either the first is false or the second is false.
Either the first is true or the second is true,

which again is equivalent to
One must be true and one must be false.

The fourth definition:

It cannot be that the first is true and the second true; nor can it be that the first is false and the second is also false: which amounts to saying that they cannot both be true and they cannot both be false. This was the definition originally given.

The student will gain familiarity with these equivalences by working out for himself the corresponding equivalent definitions for the other relations.

Similarly, we may define the seven relations of opposition in terms of a single function, say, incompatibility.

Equivalence. A proposition p is equivalent to a proposition q when it cannot be that p is true and q is false nor that p is false and q is true,

i.e. $(p|\bar{q}) \cdot (\bar{p}|q)$.

Super-alternation. A proposition p is super-altern to a proposition q when it cannot be that p is true and q is false, but it may be that p is false and q is true,

i.e. $(p|\bar{q}) \cdot \sim(\bar{p}|q)$.

Sub-alternation. A proposition p is sub-altern to a proposition q when it cannot be that p is false and q is true, but it may be that p is true and q is false,

i.e. $(\bar{p}|q) \cdot \sim(p|\bar{q})$.

Independence. A proposition p is independent of a proposition q when the truth of p is not incompatible with either the truth or the falsity of q, and the falsity of p is not incompatible with either the truth or the falsity of q,

i.e. $\sim(p|q) \cdot \sim(p|\bar{q}) \cdot \sim(\bar{p}|q) \cdot \sim(\bar{p}|\bar{q})$.

Contrariety. A proposition p is the contrary of a proposition q when it cannot be that p is true and q is true, but it may be that p is false and q is false,

i.e. $(p|q) \cdot \sim(\bar{p}|\bar{q})$.

Sub-contrariety. A proposition p is the sub-contrary of a proposition q when it cannot be that p is false and q is false, but it may be that p is true and q is true,

i.e. $(\bar{p}|\bar{q}) \cdot \sim(p|q)$.

Contradiction. A proposition p is the contradictory of a proposition q when it cannot be that p is true and q is true, and it cannot be that p is false and q is false,

i.e. $(p|q) \cdot (\bar{p}|\bar{q})$.

Corresponding definitions can be given in terms of implication, counter-implication and alternation.

In reducing the definitions of the relations of oppositions to a single principle certain characteristics of these relations, obscured by the usual definitions, are revealed. It will be noted that equivalence combines the positive features of sub-alternation and super-alternation, i.e. those contained in the clauses of the definitions which assert incompatibilities. Contradictories combine the corresponding characteristics of the contrary and the subcontrary relations. Independence combines the negative features (i.e. those asserted in clauses denying incompatibility) of super-alternation, sub-alternation, the contrary and the subcontrary relations.

From the equivalence of the four ways of defining relations of opposition it follows, further, that we can place four interpretations upon the square of opposition.[1]

If we conceive the relations in terms of disjunction the square exhibits the incompatibilities between the four forms. We may re-state the facts in a table of disjunctions:

	A	E	I	O
A true is inconsistent with	False	True	False	True
E ,, ,, ,,	True	False	True	False
I ,, ,, ,,		True	False	
O ,, ,, ,,	True			False
A false is inconsistent with	True			False
E ,, ,, ,,		True	False	
I ,, ,, ,,	True	False	True	False
O ,, ,, ,,	False	True	False	True

Similarly, interpreting the relations in terms of alternation the square of opposition exhibits the alternatives with which a consistent thinker is presented. He must either accept A or accept O. He must either reject A or reject E and I, etc. A table of alternations corresponding to the table of disjunctions may be readily constructed.

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[1] There is another sense in which the square of opposition admits of diverse interpretation, viz. according as the various forms are regarded as having existential import or not. Cf. Keynes, 'Formal Logic', Pt. II., Ch. VIII. With these differences we are not concerned here.

Most commonly the square of opposition is regarded as a formulation of relations of implication. As we have seen, there are two questions with regard to a proposition in which the logician is pre-eminently interested: What does it imply? By what is it implied? In other words: What does it prove? and, How can it itself be proved? The answers to these questions, so far as the categorical forms are concerned, may be expressed in a table of Implications and of Counter-implications constructed in the same way as the table of disjunctions. We may combine these two tables as follows:

			A	E	I	O		A	E	I	O
A	true	implies	true	false	true	false	and is implied by	true			false
E	,,	,,	false	true	false	true	,, ,, ,, ,,		true	false	
I	,,	,,		false	true		,, ,, ,, ,,	true	false	true	false
O	,,	,,	false			true	,, ,, ,, ,,	false	true	false	true
A	false	implies	false			true	and is implied by	false	true	false	true
E	,,	,,		false	true		,, ,, ,, ,,	true	false	true	false
I	,,	,,	false	true	false	true	,, ,, ,, ,,		true	false	
O	,,	,,	true	false	true	false	,, ,, ,, ,,	true			false

§ 3. Relations between Two Classes.

In the two preceding sections we were concerned with relations between two categorical propositions without regard to the interpretation of these propositional forms. The relations of opposition are not affected according as we adopt an extensional or an intensional reading.

It is instructive, however, to examine these special relations in connection with the purely extensional reading. The traditional forms, interpreted in extension, assert a relation of inclusion or exclusion between two classes. The form *SaP* asserts that the class S is wholly included in the class *P*, *SeP* that *S* and *P* are mutually exclusive, *SiP* that S is at least partially included in P, *SoP* that S is at least partially excluded from P. When a term is employed so as to refer to the whole of a class it is said to be 'distributed'. When reference is made only to a part of a class the term is said to be 'undistributed'. Thus in *All S is P* S is distributed but P undistributed. P is undistributed since what is asserted is that S coincides with a part at least of P. In *No S is P* it is asserted that the whole of S is excluded from P, and therefore the whole of P is excluded from S. Both terms are thus distributed. In *Some S is P* nothing is asserted with regard to either the whole of S or to the whole of P. Thus neither term is distributed. In *Some S is not P* S is undistributed, but P is distributed. What is asserted is that some S is excluded from the whole of P. The facts with regard to the distribution of terms in the categorical form may be summed up by saying:

The Universal forms distribute their subjects.
The Negative forms distribute their predicates.

The fact that each term may be either distributed or undistributed might suggest that eight propositional forms are required instead of only four as in the traditional scheme. This amendment was, in fact, suggested by Hamilton.[1] The eight together with the supplementary symbols introduced are shown in the following table:

	Affirmative Forms.		Negative Forms.	
S distributed **P distributed**	All S is all P	U	No S is (any) P	E
S distributed **P undistributed**	All S is (some) P	A	No S is some P	η
S undistributed **P distributed**	Some S is all P	Y	Some S is not (any) P	O
S undistributed **P undistributed**	Some S is (some) P	I	Some S is not some P	ω

The supplementary forms, however, are open to objection on various grounds.[2] At best they are unnecessary, since any information they convey may be expressed in the four traditional forms. Thus the U proposition is equivalent to the conjunction of SaP with PaS and η is equivalent to PoS. We may therefore restrict attention to the four forms A, E, I and O, and consider the relations which they imply.

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[1] 'Lectures on Logic', Lectures VIII. and XIV. Also 'New Analysis of Logical Forms'.

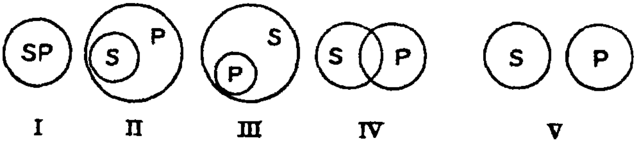
[2] For a detailed discussion see Keynes, 'Formal Logic', Pt. I., Ch. VII.

Given simply two logically independent classes S and P there are five determinate ways in which these classes may be related —ranging from complete coincidence to complete mutual exclusion:

i. The two classes may be coincident.
ii. The first may be included in the second without exhausting it.
iii. The first may include the second without being exhausted by it.
iv. The two classes may overlap, i.e. each may be partially but only partially excluded by the other.
v. The two classes may be mutually exclusive.

These relations are analogous to certain relations which may hold between closed figures in space. Hence they admit of diagrammatic representation.

Employing circles to represent classes the five relations are shown in the following diagrams:



The figures employed in this mode of representation are known as 'Euler's Diagrams'. Their use, like all thought-saving devices, is only justifiable after thought has already been employed. It is important to distinguish the logical relations from the purely spatial relations used in the representation. The fundamental notion is that of a conjunction of properties. To say that one class is coincident with another means that everything that has one property also has another. Thus the terms 'inclusion' and 'exclusion' are scarcely less figurative than the diagrams themselves.

It will be noticed that there is no simple correspondence between the four propositional forms and the five relations obtaining between classes. This results from the fact that it is not the function of the propositional forms to express determinate relations but, rather, to express *knowledge* or *belief* which may be indeterminate. A statement may be important without being determinate because it excludes certain cases which were antecedently possible. The four propositional forms may be examined from this point of view:

S*a*P is inconsistent with the cases represented in diagrams III., IV. and V. It narrows down the possibilities to the cases represented by I. and II.

S*e*P is the one propositional form which corresponds with a single diagram. It is consistent only with the case shown in V. It excludes the remaining possibilities.

S*i*P is consistent with I., II., III. and IV. and excludes only V.

S*o*P is consistent with III., IV. and V. and excludes I. and II.

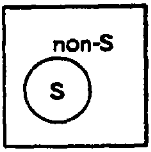
It follows that if we wish to state in the traditional forms that classes are related in one of the five determinate ways we must employ, except in the case of V., a conjunction of two or more propositions.

I. may be asserted by the conjunction S*a*P and P*a*S.
II. is asserted by the conjunction of S*a*P with P*o*S.
III. is asserted by the conjunction of P*a*S and S*o*P.
IV. requires a conjunction of three propositions S*i*P, S*o*P, P*o*S.
V. is asserted by S*e*P alone (or P*e*S alone).

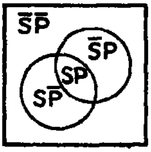
It will be noted that I. can be expressed by the supplementary form S*u*P alone as this is equivalent to the conjunction given above. The other supplementary forms do not enable us to assert determinate relations by the use of a single propositional form except in so far as we cease to employ 'some' in its ordinary logical sense. The inutility of the form *ω* is shown by the fact that it excludes no case whatever.

§ 4. Relations between Two Classes and their Contradictories.

So far, we have been concerned with the possible relations between two logically independent classes. But given any class whatever which is not co-extensive with the whole universe there will be another class to which it is necessarily related. Given the class 'animals' there is the class of things which are not animals. This is called the contradictory class. So generally: Let there be any such class, X, there will be a contradictory class, non-X. Any class together with its contradictory class will be co-extensive with the universe since everything will fall within one or the other. Many inferences consist in passing from a proposition in which a given class occurs to another proposition in which the contradictory class occurs. We must therefore amplify the general account of the relations between two classes by considering the possible relations between classes and their contradictories. The contradictory of any given class is represented diagrammatically directly we represent the universe as well as the given class. Thus, taking a square to represent the whole universe, and a circle for a given class S, the class non-S is represented by the area of the square excluded from the circle.



Given two overlapping classes, S and P, four classes result, viz. the class of things contained in both S and P, the class of things contained in S and excluded from P, the class of things contained in P and excluded from S and the class of things excluded from both. Using the symbols $\bar{S}$ and $\bar{P}$ to represent the negative classes, the four subclasses are shown as follows:



Given two classes, S and P, such that all four classes S, P, $\bar{S}$ and $\bar{P}$ are represented in the universe (i.e. presuming that neither S nor P is co-extensive with the universe), there are seven possible relations to be considered.[1] There are the five relations already considered, and in addition the two cases which result when S and P together are co-extensive with the whole universe.

The seven cases may be stated as follows:

I. The two classes are coincident, but not jointly exhaustive of the universe.

II. The first class is contained in the second, the former not being exhaustive of the latter, and the latter not being exhaustive of the universe.

III. The first class contains the second without being exhausted by it and without being itself exhaustive of the universe.

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[1] It is most important to remember this assumption throughout the whole of the following discussion. Under conditions in which this assumption is not realized a different formulation is required. The condition is commonly provided for when the two classes and their contradictories are defined by reference to an average. Cf. the concluding paragraphs of this chapter.

IV. The two classes overlap without being jointly exhaustive of the universe.

V. The two classes are mutually exclusive without being jointly exhaustive of the universe.

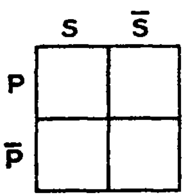
VI. The two classes overlap and are jointly exhaustive of the universe.

VII. The two classes are mutually exclusive and are jointly exhaustive of the universe.

The first five cases are those of the prior formulation, with an additional clause stating that neither one class nor the two jointly are exhaustive of the universe. This is sufficient to meet the condition that the four classes S, $\bar{S}$, P and $\bar{P}$ are represented in the universe.

The two new cases, VI. and VII., correspond respectively to cases IV. and V. They differ only in the fact that in these cases the classes are jointly co-extensive with the universe. It will be seen that this is consistent with the existence of the four classes S, $\bar{S}$, P and $\bar{P}$.

An alternative form of diagrammatic representation may be illustrated in connection with these seven relations.[1] Representing the universe by a square we may first of all divide it by a vertical line into two mutually exclusive and jointly exhaustive classes S and $\bar{S}$. Let the rectangular compartment formed on the left represent the class S, then one on the right the contradictory class $\bar{S}$. We can then divide the universe again into two contradictory classes, P and $\bar{P}$ by a horizontal line, letting the top rectangle represent the class P and the lower rectangle the class $\bar{P}$. We have thus the four resultant divisions SP, $\bar{S}$P, S$\bar{P}$, $\bar{S}\bar{P}$.



We may consider this diagram as a schema of possibilities into which different kinds and degrees of information may be fitted. We are assuming that we know only that there are members of each of the four classes S, $\bar{S}$, P and $\bar{P}$, but have no information as to how the members of these classes are distributed through the four compartments SP, $\bar{S}$P, S$\bar{P}$, $\bar{S}\bar{P}$. The

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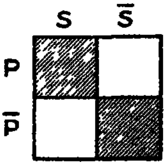
[1] They admit, of course, of representation in Euler's diagrams, but it is left as an exercise for the student to supply the necessary modifications.

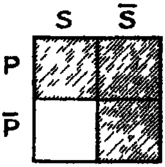
assumption is consistent with some of the compartments being empty.

We may represent the seven possibilities by shading in those cells or compartments which contain members, leaving blank those which contain no members.

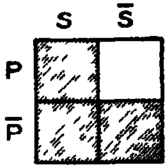
I. *Classes coincident but not jointly exhaustive.*—Since All S is P the cell $S\bar{P}$ must be empty, and since All P is S the cell $\bar{S}P$ must be empty Since there are members of the class S and of the class P both must be contained in the cell SP. Since, also, there are members of the class $\bar{S}$ and members of the class $\bar{P}$ they must both be contained in the cell $\bar{S}\bar{P}$.



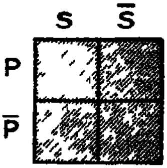
II. *S contained in P and P not exhaustive* —Since every S is contained in the class P the cell $S\bar{P}$ is empty. There are S's, so the cell SP is occupied S does not exhaust P, so the cell $\bar{S}P$ is occupied. There are $\bar{P}$'s, so the cell $\bar{S}\bar{P}$ is also occupied.



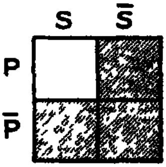
III *S contains P but does not exhaust the universe* —This case is analogous to II, with S and P changing places The only cell that is not occupied will be $\bar{S}P$.



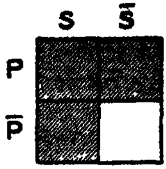
IV *The classes overlap without being jointly exhaustive of the universe* —In this case the classes SP, $\bar{S}P$, $S\bar{P}$, $S\bar{P}$ are all represented Every cell, therefore, is occupied.



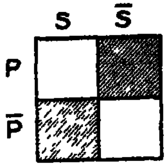
V. *The classes are mutually exclusive, but not exhaustive of the universe* —This excludes the case SP. Therefore the cell is empty The cases of S must be $S\bar{P}$, and the cases of P $\bar{S}P$. These are not exhaustive, therefore there are cases of $\bar{S}\bar{P}$.



VI. *Overlapping of classes with joint exhaustion of the universe.*—This differs from case IV. in that the class $\bar{S}\bar{P}$ is excluded. All other cells are occupied.



VII. *Classes mutually exclusive and jointly exhaustive.*—This differs from V. in the same way as VI. differs from IV. Hence the cell $\bar{S}\bar{P}$ as well as SP is empty.



We may now examine the four propositional forms to ascertain the information they yield concerning the relations between two classes and their contradictories. Determinate information is conveyed if it enables us to declare a given compartment to be occupied or to be empty. In so far as the given proposition is indeterminate we shall be left in doubt whether certain cells are occupied or not. Representing the force of the proposition diagrammatically we may proceed as before by shading in cells which are definitely asserted to be occupied, leaving blank those that are asserted to be empty. We may in addition mark with a query those cells with regard to which information is lacking.

Each propositional form excludes some of the seven possibilities enumerated above. The cells asserted to be occupied will be those which are occupied in each of the cases to which we are restricted. Those asserted to be unoccupied will be those that are unoccupied in each of these cases. Cells concerning which information is lacking will be those that are occupied in some of the cases and unoccupied in others. We obtain the following results:

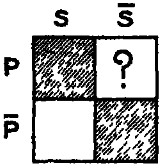
S*a*P restricts us to the possibilities I. and II. It excludes III., IV., V., VI. and VII.

S*e*P restricts us to cases V. and VII. It excludes I., II., III., IV., VI.

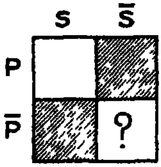
S*i*P restricts us to cases I., II., III., IV., VI., and excludes V. and VII.

S*o*P restricts the possibilities to III., IV., V., VI. and VII., and excludes I. and II.

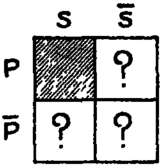
It will be noticed that the universal propositions each restrict us to two possibilities and exclude five, whereas each of the particular forms exclude only two cases, leaving five possibilities open. Further, the universal forms give determinate information with regard to three of the four cells, the particular forms giving information with regard to only one. The diagrams are as follows:



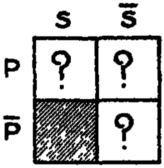
A



E



I



O

It is of importance to remember that throughout the foregoing discussion we have been concerned with the case in which it may be assumed that each of the four classes, S, $\bar{S}$, P and $\bar{P}$, are represented in the universe. This is not logically the simplest case, but it is a case of special importance. The required conditions are realized most notably when we are concerned with positive and negative classes defined with reference to a mean or average. This is very commonly the case. In a study of the relations between ability and health, for example, we might be led to examine the relations between the four classes: *able men*, *men who lack ability*, *healthy men* and *unhealthy men*. The simplest basis of classification is by reference to a mean. Able men are men above average ability. Healthy men are men who experience less than the average amount of sickness. Such definitions automatically secure that if the positive class has members, so, too, has the negative class. If, on the other hand, we were interested in the relations between ability and morality and defined 'moral men' as 'men of perfect conduct', the required conditions might fail. The class of imperfect men might exist, but what guarantee have we of the existence of perfect men?

This topic will arise for further discussion in Chapter X. It may be noted that the matters with which we have been concerned lead by an obvious extension to the statistical theory of association.

BIBLIOGRAPHICAL NOTES.

CHAPTER VI.

The doctrine of Opposition has been treated in this chapter in a manner intended to enable the reader to become accustomed to the logical relations and the notation introduced in Chapter IV. The subject is not itself controversial, though for certain purposes different interpretations of the propositional forms are desirable. For a thorough discussion of the various possible interpretations in respect of '*existential import*' *see* **Johnson, Logic,** Part I., Chapter IX., and **J. N. Keynes, Formal Logic,** Part II., Chapter VIII.

The treatment of the relations between classes should be compared with that of Keynes (Part II., Chapter V.). The method of diagrammatic representation in § 4 has been introduced in view of its convenience in the statistical developments of this subject (cf. Chapter X.). The standard treatise on these developments is **G. Udny Yule, An Introduction to the Theory of Statistics.**

CHAPTER VII.

IMMEDIATE INFERENCE.

§ 1. Obversion and Conversion.

TRADITIONALLY, inferences have been classified into the two classes 'immediate' and 'mediate'. An immediate inference is one which it is possible to draw from a given proposition alone; a mediate inference is one drawn from a given proposition through the mediation of some other proposition or propositions. The distinction, however, is an extremely fine one. If the given proposition p requires to be conjoined with q to yield the conclusion r it is clear that the so-called 'mediate' inference from p to r is equivalent to an immediate inference from the conjunctive proposition $p \, . \, q$. The distinction perhaps arises from the fact that a single proposition given externally by one speaker is sometimes sufficient to justify the required inference by another, whereas in other cases the externally given proposition has to be conjoined with some other supplied by the recipient himself in order to enable the inference to be drawn. This difference, however, does not affect the logical nature of the inference, which in both cases can be formulated as immediate.

Two classes of immediate inferences have already been considered—inferences based upon the equivalences of compound propositions, and inferences based on the relations exhibited by the square of opposition.

The inferences to be considered in the present chapter concern certain other relations obtaining between propositions of the traditional categorical form. Suppose, for example, that we are given information concerning the relations of the class of deaf-mutes to the class of persons who are short-sighted. Such information may be expressed by statements in the traditional forms interpreted in extension. It may be that the deaf-mutes are partially or wholly included in the class of persons who are short-sighted, or it may be the former are partially or wholly excluded from the latter. What can we infer as to the relation of the class of short-sighted persons to the class of deaf-mutes, and what can be inferred with regard to the relation of the class of persons who are *not* deaf-mutes to the class of persons who are *not* short-sighted? The traditional doctrine of immediate inference is mainly concerned with questions of this kind.

Stated in the technical language of this system of logical doctrine, we are concerned with the inferences from one proposition of the traditional categorical form to another proposition of the same form in which the terms either differ in position or are replaced by their contradictory terms.

There are in all eight ways in which two terms S and P together with their contradictories can be combined in a proposition of the traditional form. These are represented in the following schema:

1. S—P	5. P—S.
2. S—non-P.	6. P—non-S.
3. non-S—P.	7. non-P—S.
4. non-S—non-P.	8. non-P—non-S.

Taking (1) as the original proposition, asserting that a given subject-class, S, stands in a certain relation to a given predicate-class, P, the following are the relations we may wish to determine:

(*a*) That in which the given S-class stands to the contradictory of the given P-class, i.e. S—non-P.

(*b*) That in which the contradictory of the given S-class stands to the given P-class, i.e. non-S—P.

(*c*) That in which the contradictory of the given S-class stands to the contradictory of the given P-class, i.e. non-S—non-P.

(*d*) That in which the given P-class stands to the given S-class, i.e. P—S.

(*e*) That in which the given P-class stands to the contradictory of the given S-class, i.e. P—non-S.

(*f*) That in which the contradictory of the given P-class stands to the given S-class, i.e. non-P—S.

(*g*) That in which the contradictory of the given P-class stands to the contradictory of the given S-class, i.e. non-P—non-S.

The required inferences, which might at first sight appear to be complicated, may be reduced quite simply to two fundamental operations. The remaining inferences are all derived by combining these two. The fundamental operations are obversion and conversion, and each may be performed in accordance with very simple rules.

Obversion.—*Obversion is a process of immediate inference in which from a given proposition another is inferred having as its subject the original subject and as its predicate the contradictory of the original predicate.*

It is thus of the form S—P ∴ S—non-P.

The derived proposition is the ' obverse ' of the original, which in this connection is described as the ' obvertend '.

The obversion of the forms A, E, I and O are illustrated in the following elementary examples :

All week-days are working days, ∴ No week-days are holidays.
No months contain six Fridays ∴ All months contain less than six Fridays.
Some Sundays are wet ∴ Some Sundays are not dry.
Some years are not eventful. ∴ Some years are uneventful.

In the usual symbolism the process may be expressed as follows :

(A) All S is P, ∴ No S is non-P (E)
(E) No S is P, ∴ All S is non-P (A)
(I) Some S is P, ∴ Some S is not non-P (O)
(O) Some S is not P, ∴ Some S is non-P (I)

From a comparison of obverse and obvertend in all cases it will be seen that the obverse is obtained as a simple symbolical operation by changing the predicate term to its contradictory and changing the sign of quality. The obverse is equivalent to the obvertend since these two changes consist either in the introduction or the removal of a double negation in accordance with the principle

$$\text{not non-}p \equiv p,$$

or in transferring the negative factor from the copula to the predicate.

The full significance of the latter operation is perhaps more clearly brought out by expressing the change of ' quality ' in terms of the contrasted relations of inclusion and exclusion. In this way the apparently trivial operation of passing from *Some*

S is not P to *Some S is non-P* is seen to consist in inferring from the fact that S is at least partly *excluded* from P to the fact that it is at least partly *included* in non-P.

Conversion.—*Conversion is a process of immediate inference in which from a given proposition another is inferred having as its subject the original predicate and as its predicate the original subject.* Here the derived proposition is called the 'converse' and the original the 'convertend'. The general form of the inference is

S—P ∴ P—S.

It is obvious that in the case of the E and the I forms the process of conversion may be performed simply by reversing the positions of the terms without any other change.

No play attributed to Shakespeare was in fact written by Bacon,
∴ *No work which was in fact written by Bacon was any play attributed to Shakespeare,*

i.e. *No S is P* ∴ *No P is S.*

Similarly, if some parts of Africa have unhealthy climates, it follows that in any map of the climates of the world some unhealthy climates must be marked in Africa,

i.e. *Some S is P* ∴ *Some P is S.*

Propositions of the A and the O forms are not simply convertible. If *all great poets have great powers of imagination* it does not follow that *all who have great powers of imagination are great poets.*

So far as an A proposition is convertible at all (and it is only convertible on the assumption that there are cases of S and cases of P) it can only be converted in the manner expressed by

All S is P ∴ *Some P is S.*

In traditional terminology it is only convertible *per accidens.* It is not convertible 'simpliciter'. The principle involved is that if any term is undistributed in the convertend it must remain undistributed in the converse. This is derivative from the more general principle of inference: *given information about part only of a class we cannot infer anything concerning the whole of that class.*

The same principle contributes to the explanation of the fact that the O proposition is not convertible at all. Given

Some S is not P the converse if obtainable would be either *Some P is not S* or that *Some P is S*. The first alternative is excluded on the ground that S being undistributed in the convertend cannot be distributed in the converse. The second alternative must be excluded on the ground that the original proposition is compatible with *No S is P*, which by conversion may be seen to be incompatible with *Some P is S*.

The conditions of valid conversion are summed up in two special 'rules of conversion'.

(1) No term undistributed in the convertend may be distributed in the converse.

(2) The quality of the converse must be the same as that of the convertend.

§ 2. Derivative Forms.

All other inferences of the type we are now concerned with are simply performed by alternating the processes of obversion and conversion. Thus, having converted the given proposition we may proceed to obvert the converse so obtained. There is no distinctive name for this operation. In constructing a definition analogous to those given for the other immediate inferences, we must therefore describe it as the process of obverting the converse. The proposition obtained is called simply the obverted converse.

Obverting the Converse.—*This is a process of immediate inference in which from a given proposition another is inferred having as its subject the original predicate and as its predicate the contradictory of the original subject.* It is thus of the form

S—P ∴ P—non-S.

The three propositional forms A, E and I admit of this operation. The form O has no obverted converse since it does not admit of conversion.

Of the four cases that remain two may be described as species of Contraposition and two as species of Inversion.

Contraposition.—*Contraposition is a process of immediate inference in which the inferred proposition has as its subject the contradictory of the original predicate.*

This definition covers two cases according as the predicate is (*a*) the original subject, or (*b*) the contradictory of the original subject. These may be distinguished respectively as the 'Partial Contrapositive' and the 'Full Contrapositive'.[1]

Thus relatively to the original proposition,	S—P
we have as Partial Contrapositive,	non P—S
and as Full Contrapositive,	non P—non-S.

The partial contrapositive is obtained by converting the obverse of the original proposition. The full contrapositive is obtained by obverting the partial contrapositive. Since the obverse of E is of the A form, which is only convertible *per accidens*, it follows that the contrapositives are of the forms I and O. The obverse of I, being of the O form, is not convertible. Hence this form has no contrapositive.

Inversion.—*Inversion is a process of immediate inference in which the inferred proposition has as its subject the contradictory of the original subject.*

Here, also, two cases may be distinguished according as the predicate is the original predicate or its contradictory. Employing the same differentiating prefixes we have

The Partial Inverse	Non-S—P
The Full Inverse	Non-S—non-P.

The two inverses of a given proposition are obtained by alternately obverting and converting. It will be found, however, that in the case of the particular forms I and O the process is arrested by the inconvertibility of the O form before the inverse is reached. It will also be found that in the case of the A form the process begins with obversion whereas in the case of the E form the process begins with conversion.[2]

We may illustrate the process of inversion and the other forms of immediate inference by ascertaining the total range of the immediate inferences of this general type for each of the four forms, A, E, I and O.

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[1] These terms were introduced by Keynes. Other logicians commonly restrict the term 'Contrapositive' to what is here called the Partial Contrapositive and describe the Full Contrapositive as the 'Obverted Contrapositive'.

[2] It may assist the student to remember this rule if it be noted that in order to obtain the inverse the proposition must be preserved in universal form for as many steps as possible. In the case of the A form, if the first step be conversion the proposition immediately is reduced to particular form whereas by obversion it remains universal.

In a university, let us suppose, the regulations are such that students of Greek are compelled also to study Latin, and that students of History are precluded from the study of advanced Mathematics. Students of Botany are permitted to combine this subject with Geology and some in fact do so. Students of Chemistry are not compelled to study advanced Physics, with the result that some do not pursue the latter science beyond the elementary stages. What can we infer, in accordance with the foregoing principles of immediate inference, as to the combinations of subjects pursued by the classes of students referred to? For brevity we may write 'Hellenists' for students of Greek, 'Latinists' for students of Latin, 'mathematicians' for students of advanced Mathematics, and so on. The information given may then be expressed in the four categorical forms:

All hellenists are latinists	A.
No historians are mathematicians	E.
Some botanists are geologists	I.
Some chemists are not physicists	O.

Then, given *All hellenists are latinists*, i.e. *All S is P*, we may infer by conversion:

(1) The converse—*Some latinists are hellenists, Some P is S.*

By obverting (1) we obtain

(2) The obverted converse—*Some latinists are not non-hellenists, Some P is not non-S.*

This cannot be again converted so the series in this direction ends. Commencing again with the original proposition we may, however, infer:

By obversion

(3) The obverse. *No hellenists are non-latinists, No S is non-P.*

By converting (3) we obtain

(4) The partial contrapositive. *No non-latinists are hellenists, No non-P is S.*

By obverting (4) we obtain

(5) The full contrapositive—*All non-latinists are non-hellenists, All non-P—non-S.*

By converting (5) (*per accidens*) we obtain

(6) The full inverse: *Some non-hellenists are non-latinists, Some non-S—non-P.*

By obverting (6) we obtain

(7) The partial inverse: *Some non-hellenists are not latinists, Some non-S is not P.*

This being an O proposition it cannot be converted and the series again ends. It will be noted that in this case we obtain the full inverse before obtaining the partial inverse.

Given *No historians are mathematicians: No S is P,* we may infer by conversion:

(1) The converse: *No mathematicians are historians, No P is S.*

By obverting (1) we obtain

(2) The obverted converse: *All mathematicians are non-historians, All P is non-S.*

By converting (2) we obtain

(3) The partial inverse: *Some non-historians are mathematicians, Some non-S is P.*

By obverting (3) we obtain

(4) The full inverse: *Some non-historians are not non-mathematicians, Some non-S is not non-P.*

This being a proposition of the O form the series here ends.

Returning to the original proposition we infer by obverting

(5) The obverse: *All historians are non-mathematicians, All S is non-P.*

By converting (5) we obtain

(6) The partial contrapositive: *Some non-mathematicians are historians, Some non-P is S.*

By obverting (6) we obtain

(7) The full contrapositive: *Some non-mathematicians are not non-historians, Some non-P is not non-S.*

The series is again arrested by the inconvertibility of O.

Given *Some botanists are geologists, Some S is P,* we may infer by conversion:

(1) The converse: *Some geologists are botanists, Some P is S.*

By obverting (1),

(2) The obverted converse: *Some geologists are not non-botanists, Some P is not non-S.*

This cannot be converted.

From the original proposition we obtain by obversion :

(3) The obverse : *Some botanists are not non-geologists, Some S is not non-P.*

This again is inconvertible.

Given *Some chemists are not physicists, Some S is not P*, we cannot proceed by conversion. By obversion we obtain

(1) The obverse : *Some chemists are non-physicists, Some S is non-P.*

By converting (1) we obtain

(2) The partial contrapositive : *Some non-physicists are chemists, Some non-P is S.*

By obverting (2) we obtain

(3) The full contrapositive : *Some non-physicists are not non-chemists, Some non-P is not non-S.*

This again is inconvertible.

We have thus seven possible immediate inferences from the two universal forms and three from each of the two particular forms. These inferences may be briefly summarized in the following table. The central line gives the original proposition. The series obtained by reading upwards from the central line are those resulting from alternating conversion and obversion commencing with the former ; the series obtained by reading down from the central line are those resulting from alternating the two processes commencing with obversion. In the table non-S and non-P are abbreviated to S̄ and P̄.

TABLE OF IMMEDIATE INFERENCES.

By Obv	Full Inverse		S̄ *o* P̄		
By Conv.	Partial Inverse		S̄ *i* P		
By Obv	Obverted Converse	P *o* S̄	P *a* S̄	P *o* S̄	
By Conv.	Converse	P *i* S	P *e* S	P *i* S	
	Original Proposition	S *a* P	S *e* P	S *i* P	S *o* P
By Obv	Obverse	S *e* P̄	S *a* P̄	S *o* P̄	S *i* P̄
By Conv.	Partial Contrapositive	P̄ *e* S	P̄ *i* S		P̄ *i* S
By Obv.	Full Contrapositive	P̄ *a* S̄	P̄ *o* S̄		P̄ *o* S̄
By Conv	Full Inverse	S̄ *i* P̄			
By Obv.	Partial Inverse	S̄ *o* P			

Given the validity of obversion and conversion and the general presupposition that each of the four classes—S, P, non-S, and non-P—contain some members, the validity of these inferences is secured.

It must be noted, however, that where this presupposition does not hold good, false or absurd results will be obtained.

Reverting to the inferences with regard to classes of university students we may consider how the conclusions drawn would be affected by the further information that Greek is a compulsory subject for all students. It immediately follows that the inverse of *All hellenists are latinists is false*. This is due to the fact that the fresh information denies the existence of members of the class non-hellenists in the restricted universe of students with which we are concerned.

Similar considerations apply to inferences from propositions in which no restriction of the universe is implied. Given the proposition

No finite beings are wholly malevolent,

it would follow that *Some non-finite beings are wholly malevolent.* Formal logic does not guarantee this conclusion since it does not guarantee that all four classes of beings, finite and non-finite, malevolent and non-malevolent, exist.

In the same way we may explain the inference from *All S is P* in which P is undistributed to *Some non-S is not P* in which it is distributed. The fact that certain presuppositions are involved in these inferences shows that they do not depend solely on the given proposition, but also upon some further premiss guaranteeing the existence of the four required classes. As noted in the previous chapter the required conditions are commonly secured by defining the contradictory classes by reference to a mean.[1]

§ 3. Transversions and Other Forms of Immediate Inference.

In addition to the forms of immediate inference so far considered—inferences based on relations between compound forms, inferences based on the square of opposition and inferences consisting in, or derivative from, obversion and conversion—

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[1] Cf. p. 109.

certain other forms have received recognition in traditional logic. Many of these may be grouped under the general head of 'transversions'.[1] A transversion may be defined as an immediate inference from a proposition of one logical form to another proposition which contains the same 'material content' but is of a different logical form.

This definition covers the inference from one compound form to another. It covers also inferences from a compound proposition to one of the traditional categorical form, e.g. *If S is P then S is R* ∴ *All SP is SR*. This is also described as an inference by 'change of relation'.

It covers also transitions from a proposition of one 'order' to a proposition of a different order; e.g. *S is P is false* ∴ *S is not P*. An allied form is illustrated in the transition from a proposition of one modality to a proposition of a different modality, *S must be P* ∴ *S is P*. This has also been described as 'inference by modal consequence'. It is in certain respects analogous to inference from a superaltern to a subaltern proposition of the same form.

An important case of transversion is that of passing from one 'interpretation' of the traditional forms to a different interpretation. The commonest case of this is the passage from the predicative to the extensional reading, and the simplest case is that illustrated in the transition from '*This is round*' to '*This is a circle*'. Here two formal changes are involved. First, that which is represented by the change from the use of an adjective to the use of a general name. This verbal difference corresponds with the construction of a characterized substantive when given merely the character. Second, there is the formal change represented by the introduction of the indefinite article. This indicates that the characterized substantive is judged to be a member of the class determined by the character in question. The reverse operation is performed when what is given is that something is a member of a class, and it is inferred that it has the property by reference to which the class is defined. Since these changes entail further a change in interpretation of the copula, the process is one of considerable complexity.

In this connection, it may be noted that although the processes of obversion, conversion, etc., are applicable to the categorical forms on any of the four interpretations, they may be more complex on one of these interpretations than on another.

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[1] This term was suggested by Constance Jones in "General Logic", and has been fairly generally adopted by logicians.

Operations of transversion are frequently misdescribed in textbooks of logic as a process of 'reducing a proposition to strict logical form'. Many generations of elementary students of logic have been called upon to perform exercises which consist in re-expressing everyday statements in one of the four traditional categorical forms with a view to deducing the familiar immediate inferences. Thus to obtain, say, the full contrapositive and the partial inverse of 'A merry heart goes all the way,' the following somewhat remarkable operations are performed. The statement is first 'reduced to logical form'. In this a subject class and a predicate class must first be determined. These are presumably the class of 'merry hearts' and the class of 'things that go all the way'. As the assertion is affirmative in quality and is intended as a generalization, we obtain the A proposition:

All merry hearts are things that go all the way.

The required contrapositive and inverse which follow are:

All things that fail to go all the way are other than merry hearts. Some things that are not merry hearts are things that do not go all the way.

The preposterous nature of the resultant statements might be to some extent mitigated by a freer rendering of the original assertion, but this would transform the exercise from one of determining the logical characteristics and the implications of an admittedly metaphorical statement to one of translating the metaphors employed. The latter is not a purely logical process. It depends upon an analysis of the material content of the assertion. We may here disregard considerations relating to the literary merits of the form of the statements employed and questions concerning the practical utility of exercises of this kind, and inquire more precisely into the nature of the logical operations involved.

We may note in the first place that it is certainly incorrect to describe the process as one of 'reducing the proposition to logical form'. It is inherent in the nature of a proposition to have some logical form. It is rather a process of changing from one logical form to another. Some of these changes will be transversions of the kind already discussed. Others will be of the type briefly referred to in connection with the general question whether all propositions are of the subject-predicate form. We might now restate the question: Are all propositions, of whatever form, *equivalent to* some other proposition of the subject-predicate form? In the widest sense of the distinction between subject and predicate an affirmative answer would seem to be possible. It is always possible to distinguish a subject given for thought and what is thought about it. The subject given for thought by means of a description can always be regarded as a member of a class. What is thought about it can always be expressed in the form 'that it had a certain property'. This property defines a further class. It is upon this fact, and upon the further assumption that the subject-predicate form is the most fundamental, that the traditional exercises depend. There would seem, however, to be no warrant whatever for this latter assumption. Furthermore, there can be little doubt that the subject-predicate form on any interpretation is by no means the most suitable for many logical purposes.

BIBLIOGRAPHICAL NOTES.

CHAPTER VII.

The treatment of the traditional immediate inference here adopted is that of **J. N. Keynes, Formal Logic**, Part II., Chapter IV. On the complexity of some of the operations involved in transversion from one propositional form to another compare **Johnson, Logic**, Part I., Chapter I., § 6.

CHAPTER VIII.

THE SYLLOGISM.

§ 1. General Characteristics.

'I SHALL never make a successful politician, for I cannot remember, and that is a prime necessity of politicians'.

Few of our chains of reasoning are expressed so explicitly in syllogistic form as this; but, fully stated, a very large number would, in fact, be found to be of similar form. 'Reduced' to the standard formulæ, what is here asserted is that the whole class of good politicians is included in the class of individuals endowed with retentive memories, that the speaker is excluded from the latter class, and that in consequence he is excluded from the class of good politicians.

It is obvious that whenever we have three terms, of any type whatever, and a certain type of relation, if we are told how each of two of these terms are related to the third we can infer how they are related to one another. If two things are each equal in size to a third thing they are equal in size to one another. If the first is greater than the second, and the second greater than the third, the first is greater than the third. If A is north of B and C is south of B, A is north of C. If A is the son of B and C is also the son of B, A and C are brothers. Such possibilities of inference do not hold for all relations. If A is different from B, and C is also different from B, we can infer nothing as to the similarity or difference between A and C. They might even be identical.

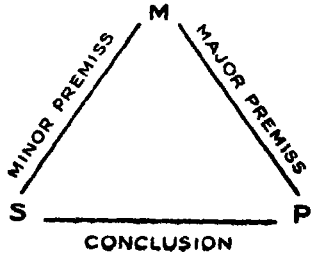
It detracts from the generality of the traditional doctrine of the syllogism that it was not clearly seen that some very general principle must cover all these cases. The natural procedure would have been to have ascertained the characters common to relations which generate these inferences and to have exhibited the doctrine of the syllogism as embracing all inferences based upon the occurrence of three terms related by a relation of the appropriate type.

The doctrine of the syllogism can be developed either as a part of the general theory of propositions, or as part of the general theory of classes. As part of the general theory of propositions it is independent of the special forms which propositions assume. We have seen that certain generalizations can be made concerning two propositions, p and q, when p implies q. The theory of syllogism is concerned with the case in which two propositions jointly imply a third, i.e. when $p . q \supset r$. This aspect of syllogistic theory will be discussed later.

If the three propositions are of the traditional categorical form and involve three terms each of which occurs in two of the propositions, syllogistic theory can be stated as a doctrine which concerns the relations between the terms. It will require to be stated differently according to which of the four interpretations is placed upon the categorical forms. The simplest account is obtained on the purely extensional reading. This, in consequence, is the most suitable to adopt in a preliminary exposition.

So interpreted, the doctrine of the syllogism belongs to the general theory of classes. It concerns the group of inferences in which a proposition asserting a relation of inclusion or exclusion (partial or complete) between two classes is inferred from two other propositions which assert respectively the relation of each of the classes to some mediating class. It will be seen that, so defined, a syllogism involves three propositions and three terms.

The term which constitutes the subject of the conclusion is called the 'Minor Term'; that which constitutes the predicate is called the 'Major Term'. The mediating term which occurs only in the premisses is called the 'Middle Term.' The premiss which states the relation of the minor term to the middle term is called the minor premiss. That which states the relation between the major term and the middle term is called the major premiss. We obtain thus the schema represented above.



Since, however, the middle term may occur either as subject or as predicate in the two premisses, it is necessary to distinguish four 'Figures':

Figure I., in which the middle term is subject in the major premiss and predicate in the minor premiss;

Figure II., in which the middle term is predicate in both premisses;

Figure III., in which the middle term is subject in both premisses;

Figure IV., in which the middle term is predicate in the major premiss and subject in the minor premiss.

When the position of the middle term is defined the positions of the major and of the minor terms become determinate in consequence. It is not necessary, therefore, to specify the positions of these terms in the definitions of the figures.

Corresponding to these definitions we obtain a schema for each figure. By convention, the major premiss is stated first.

Figure I.	Figure II.	Figure III.	Figure IV.
M—P	P—M	M—P	P—M
S—M	S—M	M—S	M—S
———	———	———	———
S—P	S—P	S—P	S—P

Syllogistic arguments are also distinguished by 'Mood'. Mood is determined by the quality and quantity of the propositions of which they are composed. The following provide examples of these distinctions of figure and mood:

1. All carnivorous animals are treacherous.
All dogs are carnivorous.
∴ All dogs are treacherous.

This is, perhaps, the commonest form assumed by a syllogistic argument. It is the only one in which a universal affirmative conclusion can be drawn. It conforms to the schema:

All M is P.
All S is M.
∴ All S is P.

The argument is in Figure I. and the three propositions are all universal affirmatives. The mood may be simply designated by the symbol AAA. It is also described as the mood 'Barbara'

—the appellation being derived from an ancient system of nomenclature devised to name the common valid moods.[1]

2. No involuntary actions are praiseworthy.
All virtuous actions are praiseworthy.
∴ Some virtuous actions are not involuntary.

The argument here is in Figure II. and the mood is EAO. It illustrates, incidentally, the significance of the terms 'weakened conclusion' and 'strengthened premiss'. The conclusion, a particular negative, might on the same premisses have been a universal negative. It is, therefore, said to be 'weakened' or in a 'subaltern mood'. Alternatively, for the same conclusion a particular affirmative minor premiss would have been sufficient. We may, therefore, say that the universal affirmative premiss actually employed is a 'strengthened premiss'. (It is not always the case that wherever the conclusion is weakened we may alternatively regard the premiss as strengthened.)

3. Some truths are dangerous.
Whatever is true should be proclaimed.
∴ Some things that should be proclaimed are dangerous.

This argument is in Figure III., the mood being IAI (Disamis). It will be noticed that even if the major premiss had been strengthened, the mood becoming AAI (Darapti), no stronger conclusion could have been drawn.

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[1] These names served as a nemonic device both for indicating the constitution of the moods and for indicating how moods in figures other than the first could be reduced to the first. The vowels indicate the propositional forms involved in the mood, the consonants, with certain exceptions, providing the necessary reminders for the procedure of reduction. These curious names were then strung into the following even more curious verses:

*Barbara, Celarent, Darii, Ferio*que prioris
Cesare, Camestres, Festino, Baroco secundae
Tertia, *Darapti, Disamis, Datisi, Felapton*
Bocardo, Ferison, habet; Quarta insuper addit
Bramantip, Camenes, Dimaris, Fesapo, Fresison.

Although these or similar verses have been learnt by heart by almost every student of Logic during the last eight centuries, it is now generally agreed that they are best forgotten. But for details of their inward significance and their history see Keynes, 'Formal Logic', and Joseph, 'Introduction to Logic'.

4. Any form of argument worthy of recognition by logic should be such as would occur in ordinary discourse.
No argument occurring in ordinary discourse is in the fourth figure.
∴ No argument in the fourth figure is worthy of recognition by Logic.

This argument, which is itself in Figure IV., is suggested by Johnson as an ironical comment upon the doctrine it contains. The mood is AEE.

Each of the four figures will be seen to involve the application of a general principle concerning the relationships of classes. These principles may be stated in four ' Dicta '.

Dictum of Figure I.

If a class M is wholly included in or wholly excluded from a second class P,
and certain objects S are included in the first class,
then these objects S are correspondingly included or excluded from the class P.

The phrase ' are correspondingly ' emphasizes the fact that if the major premiss states an inclusion the conclusion states an inclusion, and if the one states an exclusion so does the other.

Dictum of Figure II.

If a class P is wholly included in or wholly excluded from a second class M,
and certain object S are excluded from or included in the class M,
then these objects S are excluded from the class P.

It will be noticed here that the premisses state opposed relations. If one states an inclusion the other states an exclusion. The conclusion always states an exclusion.

Dictum of Figure III.

If certain objects M are included in or excluded from a class P,
and these objects M are included in a class S,
then the class S is partially included in or partially excluded from the class P.

Here the conclusion is always one of partial inclusion or partial exclusion.

These three dicta are stated so as to cover two distinct relations, the relation which holds between two classes when one is 'included in' the other, and the relation between an object and a class when the object is 'included in' the class. In the general theory of classes it is important to distinguish these relations. Traditional syllogistic theory has ignored this distinction.

In the case of the fourth figure the principle is most conveniently stated in terms of classes, and in a somewhat different form.

Dictum of Figure IV.

> Given any three classes, M, S and P, it is impossible that
> the class P should be wholly included in the class M,
> while the class M is wholly excluded from the class S,
> and the class S is partly included in the class P.

The distinctive character of this principle will be discussed subsequently.

These principles are self-evident and epistemologically independent. That is to say, each can be seen to be true independently of *prior* knowledge of the others. It will, however, be found that they are not 'constitutively' independent; i.e. it could not be the case that one might be true whilst the others were false.

It may also be observed that, in each of the first three figures, there is one proposition which is wholly 'unrestricted' as to quality and quantity. It may state either an inclusion or an exclusion and the inclusion or exclusion may be either partial or complete. This is equivalent to saying that the proposition may assume any of the four forms A, E, I or O. The two other propositions are restricted either in quality or quantity, and the restriction will be found to be either stated or implied by the special rules of the figures. In the first figure the conclusion is unrestricted, in the second the minor premiss is unrestricted and in the third the major premiss is unrestricted. In the fourth figure all three propositions are restricted: neither of the premisses can be of the form O, and the conclusions cannot be of the form A.

Since apart from special restrictions, each of the premisses and the conclusion may assume any of the four forms, A, E, I or O, there are 4 × 4 × 4, i.e. 64, possible moods in which a syllogistic argument might conceivably be cast. It is easy to see, however, that most of these moods are invalid in all figures. Of those that are valid some are valid only in certain figures. The valid moods may be determined from a few elementary rules. It will be found that there are four such rules which apply irrespective of figure, and that from these we may deduce certain general corollaries, and certain special rules for each figure. In accordance with these rules we can select the valid moods. They also serve as quasi-mechanical tests which may be applied to any argument which is cast into syllogistic form.[1]

§ 2. Rules of the Syllogism.

Traditionally, six general rules have been formulated. It will be found, however, that the first two of these are derived from the definition of the syllogism. An argument which fails to conform to them is not of necessity invalid; it is merely non-syllogistic. Of the four rules proper that remain, two are rules of Quantity and two are rules of Quality. They may, therefore, be stated as follows:

Definitions.

I. A syllogism contains three and only three terms.
II. A syllogism contains three and only three propositions.

Rules of Quantity.

I. The middle term must be distributed in at least one of the premisses.
II. If any term is distributed in the conclusion, it must be distributed in the premiss.

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[1] So far we have not been concerned with such quasi-mechanical tests, but with definitions and self-evident principles. Any clearly stated syllogistic argument can be seen to be valid or invalid by anyone of ordinary intelligence even though he cannot formulate with precision the abstract principles involved. The quasi-mechanical rules on the other hand can be applied without intelligence. The fact that certain academic exercises on the syllogism tend to encourage this procedure is

Rules of Quality.

I. The premisses must not both be negative.

II. If the conclusion is negative one premiss must be negative, and if one premiss is negative the conclusion must be negative.

Applying the definitions and rules to the criticism of arguments, we may immediately infer :

(1) If an argument contains more or less than three terms or more or less than three propositions it is not a simple categorical syllogism.

(2) If the middle term is undistributed in both premisses, or if either the major or the minor terms is distributed in the conclusion without being distributed in the premiss, the argument is formally invalid.

(3) If both premisses are negative, or if an affirmative conclusion is drawn when one premiss is negative, or if a negative conclusion is drawn when both premisses are affirmative, the argument is formally invalid.

It will be noticed that the first rule of quantity and the first rule of quality supply the tests to which the premisses must conform, whilst the second rule in both cases directs attention to the relation of the premisses to the conclusion drawn. It should also be carefully noticed that whilst the second rule of quality is symmetrical the second rule of quantity is not. If the conclusion is negative one premiss must be, and *vice versa.* If a term is distributed in the conclusion, it must be distributed in the premiss ; but the converse does not hold.

Breaches of the rules of quantity have received distinctive names. Failure to conform to the first rule of quantity constitutes the fallacy of ' undistributed middle term '. According as the offending term is the major or the minor, breach of the second rule is described as an ' illicit process of the major term ' or as an ' illicit process of the minor term '.

The peculiar technicalities of expression in these rules are not conducive to self-evidence. Restricting ourselves to the purely extensional reading of propositional forms the rules may be more naturally expressed in the following way :

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what, perhaps more than anything, has brought this part of logical theory into not wholly undeserved disrepute. The following section should be studied with this in mind.

Rules of Quantity.

I. No conclusion can be drawn concerning the relation between two classes if the evidence refers only to a part of some third class to which both are related.

II. Given evidence concerning a part only of any class no conclusion can be drawn with regard to the whole of that class.

Rules of Quality.

I. No conclusion can be drawn concerning the relations between two classes if the premisses concern only relations of exclusion (partial or complete).

II. If one of the premisses states a relation of exclusion, the conclusion also must state an exclusion, and *vice versa.*

From the general rules the following three corollaries, among others, may be deduced :

I. No conclusion can be drawn from two particular premisses.

II. If one premiss is particular the conclusion must be particular.

III. If the major premiss is particular the minor premiss must be affirmative.

Proof of the First Corollary.

There are three possible cases : (*a*) Both premisses are affirmative, (*b*) Both are negative, and (*c*) One is affirmative and the other negative.

Case (*a*). Both premisses being particular affirmative, i.e. of the form I, no term is distributed. Hence the middle term is undistributed.

Case (*b*). Directly excluded by the first rule of Quality.

Case (*c*). One premiss being particular affirmative (I), and the other particular negative (O), they distribute only one term between them, viz. the predicate of the particular negative. By the second rule of Quality the conclusion must be negative, and hence the major term is distributed in the conclusion. By the second rule of Quantity this must be distributed also in the major premiss. Since the middle term must also be distributed two terms must be distributed in all. This combination of premisses thus distributes one term less than is required by the general rules.

Proof of the Second Corollary.

The same three cases require to be considered.

Case (*a*). Both premisses being affirmative, one must be A the other I. These distribute between them only one term. This must be the middle term. Neither major nor minor being distributed in the premisses, neither can be distributed in the conclusion. It is in consequence particular.

Case (*b*). Both premisses being negative, this is directly excluded by the general rules.

Case (*c*). One premiss affirmative and one negative. The possible combinations, EI and AO, irrespective of order, distribute in both cases only two terms. One must be the middle term ; the other must be the major term, since the conclusion must be negative. The minor term thus being undistributed in the premiss must be undistributed in the conclusion.

Proof of the Third Corollary.

There are two cases (*a*) the Major premiss affirmative, (*b*) the Major premiss negative.

Case (*a*). If the major is particular affirmative it distributes neither of its terms. The conclusion must thus be affirmative, since if negative the major term would be distributed in the conclusion but not in the premiss. If the conclusion is affirmative, both premisses must be affirmative by the second rule of Quality. Hence the minor premiss is affirmative.

Case (*b*). If the major is particular negative the minor must be affirmative by the first rule of Quality.

From the rules and the corollaries the valid moods may be readily deduced. There are sixteen possible ways in which the four forms can be paired in the premisses. Of these eight are precluded by the rules and corollaries. Eight possibilities thus remain. These are shown in the following table in large type, the excluded cases being shown in small type :

AA	*EA*	*IA*	*OA*
AE	EE	IE	OE
AI	*EI*	II	OI
AO	EO	IO	OO

Given these eight possible pairs of premisses we can next determine the conclusions which the rules permit. It will be found that the three pairs of universal conclusions admit each of two conclusions and that the remainder admit of one only. Thus of the sixty-four possible combinations there are only eleven valid moods :

AAA
AAI
AEE
AEO
AII
AOO
EAE
EAO
EIO
IAI
OAO

The simplest procedure for determining the figures in which these moods may occur is by the formulation of special rules. The following are the special rules as commonly stated :

Figure I.

Special Rules.

M—P (1) The Minor premiss must be affirmative.
S—M (2) The Major premiss must be universal.

S—P *Valid Moods* : AAA, AAI, AII, EAE, EAO, EIO.

The proof of the first rule is as follows : If the minor premiss were negative, the conclusion would be negative. In which case the major term would be distributed in the conclusion. It would then require to be distributed in the major premiss, which in consequence would also require to be negative. Thus if the minor premisses were negative both premisses would require to be negative. Hence the minor must be affirmative.

The second rule is consequential on the first. The minor premiss being affirmative the middle term is undistributed in that premiss. It must, therefore, be distributed in the major premiss. This requires the major premiss to be universal.

Figure II.

Special Rules.

P—M (1) One premiss must be negative.
S—M (2) The Major premiss must be universal.

S—P *Valid Moods :* AEE, AEO, AOO, EAE, EAO, EIO.

The first rule is required to secure distribution of the middle term. The second rule is again consequential, being required as a guard against illicit process of the major term.

Figure III.

Special Rules.

M—P (1) The Minor premiss must be affirmative.
M—S (2) The conclusion must be particular.
S—P *Valid Moods:* AAI, AII, EAO, EIO, IAI, OAO.

Rule (1) follows from the same consideration as pertains to Figure I.; rule (2) is again consequential.

Figure IV.

Special Rules.

P—M (1) If the major is affirmative the minor must be universal.
M—S
S—P (2) If either premiss is negative the major must be universal.
(3) If the minor is affirmative the conclusion must be particular.

Valid Moods: AAI, AEE, AEO, EAO, EIO, IAI.

Here, the requirements cannot be stated conveniently in less than three rules.[1] The first rule secures distribution of the middle term. The second guards against illicit process of the major term. The third guards against illicit process of the minor term.

Certain distinctive features of the four figures may be noticed. In Figure I. alone is it possible to prove conclusions of all four forms, A, E, I, O. It is also the only figure in which a universal affirmative conclusion can be proved. It is consequently the most important and the most generally useful of the figures. It is also the only figure in which both major and minor terms occupy the same place in the premisses as in the conclusion. It is in consequence the simplest and the most natural form of syllogistic reasoning. Such considerations, perhaps, explain why Figure I. has been regarded as the most fundamental.

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[1] For mnemonic purposes the student may note that the third rule combines into a single implicative proposition the two rules of Figure III. The second rule similarly combines the two rules of Figure II. The first rule is *analogous to* a similar combination of the rules of Figure I., but with the modification required by the difference in the position of the terms.

Figure II. proves only negative conclusions. It is described by Whately [1] as the figure of exclusion, since it may be employed to determine the character of something by progressive elimination of possibilities. Certain symptoms are distinctive of small-pox. The patient lacks these symptoms, therefore he has not got small-pox. Certain other symptoms indicate sciatica. The patient lacks these symptoms, therefore he has not got sciatica. And so on.

Figure III. proves only particular conclusions and is therefore suited to the disproof of generalizations. On the other hand, it is also described as the Inductive Figure since in disproving that two properties are incompatible it may provide the first step in a proof that they are invariably connected. It is the most natural figure to employ when the middle term is singular. *Leibnitz was a mathematician. Leibnitz was also a great philosopher.* The conclusion to be drawn might serve not only as a disproof that philosophical and mathematical ability are incompatible, but also as a first step in an attempt to show that they are essentially connected.

Figure IV. stands in several respects in contrast with the first three figures. It was not discussed by Aristotle and many logicians have refused to 'recognize' it. But that it is one of the logical possibilities is undeniable. It must be included in any systematic account of these possibilities. It is true, nevertheless, that many arguments do not fall naturally into this form. Johnson no doubt had to exercise deliberate contrivance in the design of the illustration cited.

The 'unnaturalness' of the fourth figure arises primarily from the 'predicative reading'. In the argument:

> Every criminal is intelligent in matters pertaining to his own interest,
> Everyone who is intelligent in matters pertaining to his own interest is honest,
> ∴ Some honest men have criminal characteristics,

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[1] 'Elements of Logic'.

each term appears once as a class concept and once as a predication. Consequently the movement of thought is complicated by the necessity for a series of transversions of the kind required in passing from one reading of the traditional form to another.[1] On the purely extensional reading the premisses are such as do not require these transversions. The propositions in fact are similar to equations which can be read indifferently from left to right or from right to left. Symbolizing the relation of being exactly co-extensive with by = we could construct a syllogism of the form

$$\begin{array}{l} M = P \\ S = M \\ \hline S = P \end{array}$$

Reading this in the ordinary way we obtain a syllogism in Figure I. But if we read the two premisses from right to left and the conclusion from left to right the syllogism is in Figure IV. The fact that an ordinary syllogism does not always admit of reading in alternative directions results merely from the asymmetrical character of the relations involved.

§ 3. The Dictum de Omni et Nullo and the Principles of Reduction.

The rules of the syllogism are ill-adapted to exhibit the fundamental principles upon which syllogistic reasoning depends. As we have already noted, they are primarily a set of *criteria* by the use of which the validity of an argument may be almost mechanically tested. Nevertheless, they are logically interdependent, and it is possible to select a single self-evident rule from which the rest may be deduced. The most convenient for this purpose is the second rule of quantity: If information be given with regard to a part only of some class no conclusion can be drawn concerning the whole class.[2]

In the same way the positive principles for the several figures are logically interdependent. This interdependence can be exhibited in more than one way. Traditionally the procedure has been to 'reduce' arguments in Figures II., III. and IV. to arguments in Figure I. It was supposed that the principle of Figure I. was epistemically and constitutively prior to any principles which could be formulated for the other figures. In consequence it was possible to be assured of the validity of the latter by demonstrating their equivalence to the former.

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[1] Cf. p. 121

[2] The proof of other rules from this is given by Johnson, 'Logic', Vol. II., Ch. IV., § 11.

The principle of Figure I., originally stated by Aristotle, is known as the *dictum de omni et nullo*, on the ground that it constitutes an axiomatic truth concerning all or none of a class. It is generally stated in some such form as the following :

> *Whatever is predicated affirmatively or negatively of every member of a class, may be predicated in like manner of everything asserted to be contained in that class.*[1]

It will be seen that this principle provides for three terms : (1) a *class* concerning the whole of which something is predicated (M) ; (2) that which is predicated of the class (P) ; (3) that which is asserted to be contained in the class (S). It provides also for three propositions : (1) One predicating something of M (the major premiss) ; (2) one asserting something to be contained in this class (the minor premiss), and (3) one which predicates of the minor term whatever is predicated the major (the conclusion). The dictum also provides the four rules. It secures that the middle term shall be distributed by prescribing that what is predicated of the class M shall be predicated of every member of this class. It provides against illicit process of the major term by prescribing that what is predicated of M shall be predicated 'in like manner' of S. The phrase 'in like manner' refers to the quality of the proposition. It implies, therefore, that if the major premiss is affirmative the conclusion must be negative, and if the major is negative the conclusion must be negative. It will be seen that under these conditions there can be no illicit process of the major term. Illicit process of the minor term is provided against inasmuch as the dictum allows us to predicate P only of what has been asserted to be contained in M, i.e. a part or the whole of S. The possibility of two negative premisses is precluded by the assertion that the minor premiss must assert that something is contained in M. The second rule of quality is also provided for by the assertion that the final predication must be 'in the manner' of the first.

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[1] This dictum differs from that previously stated for Figure I. in being expressed in terms of the predicative reading of categorical forms.

Arguments in figures other than the first are brought within the scope of the dictum by the procedure of reduction. This may be direct or indirect.

Direct Reduction is effected by the operation of conversion. Clearly, given an argument in Figure II. of the form, No P is M; All S is M, .·. No S is P we have only to convert the major premiss to obtain a syllogism in Figure I. In some cases it may be necessary also to transpose the premisses:

$$\begin{tikzcd}
\text{All P is M} \arrow[dr] & \text{No S is M} & \equiv \text{No M is S} \\
\text{No S is M} \arrow[ur] & \text{All P is M} & \equiv \text{All P is M}
\end{tikzcd}$$

$$\begin{array}{ccc}
\hline
\text{No S is P} & \text{No P is S} & \text{No P is S} \\
 & & \equiv \text{No S is P}
\end{array}$$

By introducing obversion, as well as conversion, not only can any mood in any other figure be reduced to *some* mood in Figure I., it can be reduced to *any* mood of that figure (other than the subaltern moods).[1]

Indirect Reduction is a procedure whereby an argument in Figure I. is employed to prove that the conclusion in a syllogism in another figure is the only valid one. If the conclusion of any syllogism is invalid then the contradictory of that conclusion will be consistent with the premisses. Let us suppose, then, that doubt attaches to the conclusion of a syllogism in the mood AOO in Figure II. The form is

All P is M.
Some S is not M.
.·. Some S is not P.

If the syllogism is invalid, the contradictory of this conclusion is consistent with the premisses. But combining the contradictory of the conclusion with the first premiss, we obtain, in Figure I.,

All P is M.
All S is P.
.·. All S is M.

This conclusion must be false, since it is inconsistent with *Some S is not M*, which was given as true. We have thus shown by a *reductio ad absurdum*, using only Figure I. (and the principle that two contradictory propositions cannot both be true), that *Some S is not P* is validly deduced in the argument in Figure II. This is the principle of indirect induction.

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[1] A systematic account of these processes is to be found in Keynes' 'Formal Logic', Pt. III., Ch. III.

The principles of reduction were of fundamental importance so long as it was supposed that the *dictum de omni* was the sole foundation of syllogistic reasoning. Since, however, the dicta of the other figures are now seen to be equally fundamental, the principles of reduction are only of secondary interest as exhibiting relations of equivalence between syllogistic forms.

The theory of direct reduction is generalized by extending it to include all the immediate inference based upon conversion and obversion, and covers the procedure whereby a syllogism in one figure is reduced to another figure, or from one mood to another mood in the same figure.

Indirect reduction, on the other hand, is a special application of the general doctrine of the *antilogism*. This doctrine arises from an extension of the principles relating to compound propositional forms to cases involving more than two propositions.

§ 4. The Antilogism.

The principle of the syllogism is a statement to the effect that two propositions jointly imply a third, i.e. it is of the form : *If p and q, then r.* Now, as the implicative proposition *If p then q* is equivalent to a counter-implicative *If $\bar{q}$ then $\bar{p}$*, the alternative *Either $\bar{p}$ or q*, and to the disjunctive *Not both p and $\bar{q}$*, so we obtain four corresponding forms for the syllogism :

If p and q then r
$\equiv$ If $\bar{r}$ then not both p and q
$\equiv$ Either not-(p and q) or r
$\equiv$ Not p, q and $\bar{r}$.

The last form is described as an antilogism. In the most general sense of the term an antilogism is simply a formal disjunction of two or more propositions which are merely hypothetically entertained. We assert an antilogism when we assert that two, three, four or any number of propositions cannot be true together.[1]

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[1] This is Johnson's definition (' Logic ', Vol. II., Ch. IV., 3). The term, however, was introduced by Mrs. Ladd-Franklin who described a disjunction of three propositions as an ' inconsistent triad ' and an antilogism as an inconsistent triad in which the propositions contained three and only three terms. (See Baldwin's ' Dictionary of Philosophy ' ; Art. ' Symbolic Logic '.)

Given the disjunction of p, q and r we may infer:

(1) If p and q then $\bar{r}$.
(2) If p and r then $\bar{q}$.
(3) If q and r then $\bar{p}$.

Next, let p, q and r stand for the three propositions:
$p =$ Every member of a class has a certain property. (All A is B.)
$q =$ A certain object is included in that class. (C is A.)
$r =$ This object lacks that property. (C is not B.)

We obtain the inferences:

(1) If p and q then $\bar{r}$, i.e. If All A is B and C is A, then C is B.
(2) If p and r then $\bar{q}$, i.e. If All A is B and C is not B, then C is not A.
(3) If q and r, then $\bar{p}$, i.e. If C is A and C is not B, then Some A is not B.

The first case is an argument in Figure I., the second in Figure II., and the third in Figure III. The equivalence of these moods is thus demonstrated.

Employing the same principles, we may exhibit the valid moods of Figure IV. as issuing from a similar antilogism. The principle of Figure IV. has already been stated explicitly as a disjunction of three propositions. Given that it is impossible that the three propositions can be true together: p (All A is B), q (No B is C), and r (Some C is A), we obtain:

(1) If p and q then $\bar{r}$, i.e. If All A is B and No B is C, then No C is A.
This is in the mood AEE.
(2) If q and r then $\bar{p}$, i.e. If No B is C and Some C is A, then Some A is not B.
This is in the mood EIO.
(3) If r and p then $\bar{q}$, i.e. If Some C is A and All A is B, then Some B is C.
This is in the mood IAI.

The remaining moods of Figure IV. are obtained respectively by weakening the conclusion of AEE, by strengthening the minor premiss of EIO and strengthening the major premiss of IAI.

The antilogism serves to illustrate the distinction between the syllogism as a principle of *implication* and the syllogism as a principle of *inference*. A principle of implication asserts merely that a certain proposition implies another. It does not assert that the implying proposition is, in fact, true. For inference we require not only the assertion of implication, but also the assertion of the implying proposition. To employ a syllogism for inference we must know not only that the premisses imply the conclusion but also that the premisses are true. It may therefore be stated explicitly in the form :

If p and q then r,
p and q,
$\therefore r$.

When, as is commonly the case, the propositions p, q and r are each to some extent doubtful, the process of argument often exhibits an operation known as ' reversing the antilogism '. This consists in meeting an implicative syllogistic argument by its counter-implicative. Two disputants, A and B, agree that p, q and r cannot be true together. A argues : Since p and q are the case, r must be false. B, who is convinced of the truth of r, replies : Since r is true, either p or q must be false. B's position is, in general, stronger than A's, since he commits himself to only one dubious assertion as against A's conjoint assertion of two. In fact, B's position may be stronger, even though r is the least probable of the three propositions. If the chance that p is true is one in two and the chance that q is true is also one in two, whereas the chance that r is true is only one in three, it follows that the chance that r is true is greater than the chance that *both* p and q are true—the latter being only one in four.

An ingenious example is cited by Johnson [1] from the history of philosophy. An historic controversy respecting the origin of the conception of a 'substance' may be represented in the following antilogism :

It cannot be the case that
(1) All possible objects of thought are sense-given.
(2) ' Substance ' is a possible object of thought.
(3) Substance is not sense-given.

Johnson suggests that the acceptance of (1) and (2) and the consequent rejection of (3) is characteristic of certain forms

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[1] ' Logic ', Vol. II., Ch. IV., 5.

of 'realism'; that the acceptance of (1) and (3) with the rejection of (2) represents very fairly the position of Hume; and that the acceptance of (2) and (3) to the exclusion of (1) represents the position of Kant.

§ 5. Compound Arguments.

The traditional categorical syllogism has been seen to be equivalent to the conjoint assertion of an antilogism and of two of the conjoined simple propositions. It is thus a special case among an immense variety of possible forms of argument which differ (*a*) in the number of propositions disjunctively combined, and (*b*) in the number of components simultaneously asserted. Each of these forms will admit of equivalent statements derived by substituting for the disjunctive proposition the corresponding implicative, counter-implicative or alternative propositions. Moreover, many other forms are conceivable in which the second premiss is in itself an implicative, counter-implicative, alternative, or disjunctive proposition. Some of the simpler of these cases have received recognition in traditional logic. They are described as 'The Compound Modes', the 'Hypothetical' and the 'Disjunctive' Syllogism, the 'Dilemma' and by other special titles. A few of the more important forms will be briefly outlined.

The Compound Modes.—The simplest cases are those consisting in the conjoint assertions of an implicative or other compound proposition together with the assertion of one of its components, or with the assertion of the contradictory of one of its components. Arguments in this form are described as 'compound modes'. Consider the following cases:

1. The experts assert that if the revolver was found more than twelve feet distant from the body, the victim of the fatality must have been shot by some one other than himself.

Witness A testifies that the revolver *was* more than twelve feet from the body. This, however, is challenged by witness B, who says that the revolver *was not* more than twelve feet away.

Here the expert's assertion is of the form *If p then q*. Witness A asserts p; witness B asserts $\bar{p}$.

Combining *If* p *then* q with the assertion of p we may infer q, i.e. that the victim was murdered. Combining *If* p *then* q with $\bar{p}$ we cannot assert that the victim was murdered. Nor, on the other hand, can we deny it. Murder is consistent with both positions of the revolver.

2. The medical witness testifies that if arsenic had been placed in the deceased's wine he would have died before midnight. Witness C asserts the deceased was alive at dawn the following morning. Witness D contends, on the other hand, that death occurred before midnight. Hêre the implicative *If* p *then* q may be combined with $\bar{q}$ (C's assertion); in which case we may infer $\bar{p}$, i.e. that arsenic was not placed in the wine. The evidence of witness D combined with the implicative statement, on the other hand, yields no conclusion. Death might have been due to other causes.
3. Dr. P and Dr. Q are in partnership and so are precluded from taking their holidays at the same time. One or other must be on duty. We may therefore argue from the fact that one is on holiday the other is on duty. The form is: *Either Dr. P is on duty or Dr. Q is on duty. Dr. P is not on duty, ∴ Dr. Q is.* We may in the same way prove Dr. P to be on duty by showing that Dr. Q is off duty; but, of course, the fact that one is on duty does not prove that the other is on holiday.
4. X and Y are members of the same club, but cannot abide each other's company. If one enters a room occupied by the other the latter goes out. Hence, we may assert: *Not both X is present and Y is present. On such and such an occasion X was present, ∴ Y was absent.* The affirmation of one of the disjuncts enables us to deny the other. The case is distinguished from the former in that the joint presence is precluded. Joint absence is here possible.

The general nature of these inferences becomes obvious once the forms of the constituent propositions are clearly seen. It remains only to note the traditional technical descriptions. They are described by terms derived from the Latin verbs *ponere* (to affirm) and *tollere* (to deny). The gerunds *ponendo* (by affirming) and *tollendo* (by denying) indicate the nature of the simple premiss; and the participles *ponens* (affirming) and *tollens* (denying) indicate the conclusion. We obtain thus four ' Modes '.

1. *Modus Ponendo Ponens. If* p *then* q*, and* p ; $\therefore q$, i.e. the mode in which by affirming (the antecedent) we affirm (the consequent). It is matched by a corresponding fallacy—the fallacy of denying the antecedent with the object of denying the consequent.
2. *Modus Tollendo Tollens. If* q *then* p*, and* $\bar{p}$; $\therefore \bar{q}$, i.e. the mode in which by denying (the consequent) we deny (the antecedent). It, too, has its corresponding fallacy—the fallacy of affirming the consequent and inferring therefrom the truth of the antecedent.
3. *Modus Ponendo Tollens. Not both* p *and* q*, and* p ; $\therefore \bar{q}$. This is the modus in which by *affirming* (one disjunct) we are compelled to *deny* (the other disjunct). Its corresponding fallacy might be described as the fallacy of ' denying a disjunct ', where the intention is to infer the truth of the other disjunct.
4. *Modus Tollendo Ponens. Either* p *or* q*, and* $\bar{p}$; $\therefore q$, i.e. the modus in which by denying (one alternant) we are led to affirm (the other alternant). The corresponding fallacy is that of ' affirming an alternant ' where there is an intention to prove the falsity of the other alternant.

The rules for these inferences are obvious :

Ponendo Ponens. By affirming the implicans of an implicative proposition we may infer the truth of the implicate.

Tollendo Tollens. By denying the implicate of a counterimplicative proposition we may infer the falsity of the implicans.

Ponendo Tollens. By affirming one disjunct of a disjunctive proposition we may infer the falsity of the other disjunct.

Tollendo Ponens. By denying one alternant of an alternative proposition we may infer the truth of the other alternant.

The Dilemma.—Among the more complex forms of compound arguments one of the most interesting is the dilemma. It may be technically defined as an argument in which the ' major ' premiss is a conjunction of implicative propositions and the 'minor' premiss an alternative combination either of the antecedents of the two implicatives or of the contradictories of their consequents. When the minor affirms one of the antecedents the dilemma is said to be *constructive;* when it denies one or other of the consequents it is said to be *destructive*.

Constructive Form.	Destructive Form.
If p then q and if r then s.	If p then q and if r then s.
Either p or r.	Either $\bar{q}$ or $\bar{s}$.
$\therefore$ Either q or s.	$\therefore$ Either $\bar{p}$ or $\bar{r}$.

For example:

> If I back the horses at the races I lose my money, and if I don't back them I am bored; so, as I must back them or abstain from doing so, I have to choose between losing my money and being bored.

This is a constructive dilemma. As an example of the destructive form:

> If I am to please George I must go to the races with him; and if I am to please Uncle Henry I must go to the agricultural show. I cannot both go to the races and go to the show, so I am bound to offend either George or Uncle Henry.

It will be noticed that in the destructive dilemma the minor premiss is equivalent to a disjunction of the consequents.

If in the constructive form the two implicatives have a common consequent, or if in the destructive form they have a common antecedent the dilemma yields a simple categorical conclusion.

Thus:

> If I am bored I shall be irritable, and if I lose my money I shall be irritable. At the races I shall be either bored or lose my money. So I shall be irritable in any case. Moreover:
>
> If I go to the races I shall come home miserable. And, furthermore, if I go to the races I shall also offend Uncle Henry. I am not prepared both to offend Uncle Henry and to make myself miserable. Therefore I will not go to the races.

Although it is an entirely non-logical consideration we may notice that the conclusion of a dilemma is usually unpleasant.

Hence, perhaps, the popularity of this mode of argument as a dialectical weapon. Hence, too, the attention which has been devoted to the procedure for 'rebutting a dilemma'. One technical device is the following. The dilemma may be 'rebutted' by substituting the contradictory for each consequent in the major premiss and changing their places.

Thus given:

If p then q, If r then s.
Either p or r.
$\therefore$ Either q or s.

We may often argue:

If p then $\bar{s}$, If r then $\bar{q}$.
Either p or r.
$\therefore$ Either $\bar{s}$ or $\bar{q}$.

An historical example is the following.[1] An Athenian mother advised her son not to enter politics, reasoning thus:

> If you act justly men will hate you. If, on the other hand, you act unjustly the gods will hate you. Since you must act either justly or unjustly you will be hated either by men or by the gods.

The son retorts:

> If I act justly the gods will love me. If I act unjustly men will love me. Hence (on the same minor premiss) I shall be loved either by the gods or by men.

The retort, of course, does not disprove the original argument, since the two conclusions are consistent. The procedure would seem to be merely a formal device for looking on the bright side of things.

Polysyllogisms.—Arguments may be composed of chains of syllogisms, the conclusion of one providing a premiss for the next. In such a chain a syllogism the conclusion of which provides a premiss for another syllogism is called a *pro-syllogism.* A syllogism in which a premiss is provided by a pro-syllogism is called an *epi-syllogism.*

Consider the following argument:

The feeble-minded are outwitted in the struggle for existence, and those who are so outwitted suffer privation. Privation engenders bitterness, and in consequence, fosters opinions subversive of the established order of society. Those so indoctrinated become revolutionaries. Since, moreover, the revolutionaries are makers of history (for whatever makes for change makes history), and since, as we have seen, the feeble-minded become revolutionaries, it follows that the course of history depends in no small degree upon the feeble-minded.

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[1] From Welton and Monahan, 'Intermediate Logic', p. 203.

The final stage of this argument is of the form :

All revolutionaries are makers of history (because All who make for change make history).
The feeble-minded are revolutionaries.
∴ The feeble-minded make history.

This is a syllogism in the mood Barbara with the addition of a supporting proposition for the major premiss. It is thus what is technically described as an *epicheirema*. Examining the relation of the major premiss to its supporting proposition it will be seen that these two propositions constitute a syllogism in which one premiss is suppressed. Presumably the premiss is suppressed because it is too obvious to require explicit mention. Fully rendered it becomes :

All who make for change make history.
(Revolutionaries make for change.)
∴ Revolutionaries make history.

This is known as an *enthymeme*. An enthymeme is a syllogism in which one premiss is not explicitly stated. It is said to be of the first order if the major premiss is suppressed, of the second order if the minor is suppressed, and of the third order if the conclusion is suppressed. It follows that an epicheirema may be defined as a syllogism containing a premiss which is the conclusion of an enthymeme. Epicheiremas are 'single' or 'double' according as one or both premisses have supporting propositions.

Turning to the minor premiss of the principal argument, we find that it is supported by a chain of reasoning which can be rendered as follows :

All feeble-minded persons are outwitted, etc.
All who are outwitted, etc., suffer privation.
All who suffer privation are embittered.
All who are embittered adopt certain opinions.
All who hold these opinions are revolutionaries.
∴ All feeble-minded persons are revolutionaries.

Such a chain constitutes a ' Sorites '. A Sorites is defined as a polysyllogism in which all the conclusions except the last are omitted, the premisses being given in such an order that two successive propositions contain a common term. It may easily be seen that these chains admit of analysis in accordance with this definition Thus, the chain conforms to the schema :

All A is B,
All B is C,
All C is D,
All D is E,
.·. All A is E

This is equivalent to :

All A is B,
All B is C,
(.·. All A is C),
All C is D,
(.·. All A is D),
etc

If the premisses are read downward, the argument is said to constitute an Aristotelian Sorites, if they are read upwards the argument is said to be a Goclenian Sorites. Read in the first sense the suppressed conclusions appear as minor premisses in the next step , read in the second sense the suppressed conclusion appear as the major premiss of the next step The distinction, however, is wholly devoid of logical importance [1]

The full argument thus consists in an epicheirematic syllogism in Barbara, the major premiss of which is the conclusion of an enthymeme, and the minor premiss of which is the conclusion of an Aristotelian Sorites.

BIBLIOGRAPHICAL NOTES

CHAPTER VIII

The condensed account of the Syllogism contained in this chapter should be supplemented by reference to some of the many excellent treatises in which the doctrine is treated in detail. **J. N. Keynes, Formal Logic,** must again be mentioned as providing the clearest formal exposition along conservative lines. **Johnson, Logic,** Part II., Chapter IV., should be studied for the originality, economy and elegance with which this threadbare subject is presented For an exposition of the Aristotelian doctrine the reader must again be referred to **Joseph, Introduction to Logic.**

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[1] The so-called Aristotelian Sorites does not appear in Aristotle s works The Goclenian form was formulated by Rudolf Goclenius of Marburg at the end of the sixteenth century It is the better formulation, since it corresponds to the order of the premisses adopted in the ordinary syllogism

The theory of the Antilogism was first formulated by **Mrs. Ladd-Franklin.** (See Art. '**Symbolic Logic,**' **Baldwin's Dictionary of Philosophy and Psychology.**)

Among the important topics for the discussion of which space has not been found in this chapter is the question whether the syllogism inherently involves the fallacy of *petitio principii* (begging the question). This question is discussed by most authors (See **Joseph,** Chapter XIV. ; **Stebbing,** Chapter XII , **Mill,** Chapter III Book II of the **System of Logic,** should be read together with **Johnson's** comments, **Logic,** Part II , Introduction.)

CHAPTER IX.

THE GENERAL THEORY OF DEDUCTION.

§ 1. Logic as a Deductive System.

THE system of logic outlined in the four preceding chapters has been the object of destructive criticism from various points of view. We saw in Chapter I., that the subject may be studied with one or other of four purposes in mind. Logic may be conceived as a practical aid to thought, as a theory of scientific method, as a department of the general theory of knowledge, and as a science of ' pure form '. The lines of criticism to which traditional logic is subjected will be found broadly to correspond with these diverse pre-occupations.

Traditional logic is condemned as wholly inadequate as a guide to rational and efficient thought. Its analysis of thinking is psychologically defective, and its principles provide no effective defence against the fallacies and pitfalls to which the human mind is prone. Regarded as a complete organon of scientific method, the traditional doctrine is no less glaringly inadequate. Scientific research does not consist in immediate inference, nor in syllogistic reasoning. The traditional doctrine provides no account of the basis of rational generalization, and even where formal operations are important in scientific thought, they assume a form very different from those of the traditional formulæ. Even as a philosophical science, traditional logic is defective. Its fundamental concepts lack precision, and are in many ways confused. Finally, the system is attacked from a purely formal standpoint. It is defective in the analysis of propositional forms, and its principles are not sufficiently generalized. In the present chapter we shall be chiefly concerned with the latter considerations. In the following chapter, the practical and scientific limitations of traditional logic will arise for discussion.

From the complex pattern of tendencies in the history of logical thought, one strand may be selected which we may describe as the process of formalization. It has been seen that inference has a two-fold character; it has 'epistemic' and 'constitutive' conditions. The validity of an inference depends in part upon the nature of our thinking, and in part upon the nature of what is thought about. In what is thought about we can distinguish 'matter' and 'form'. Logic has consistently tended to ignore what is purely 'material' and to concentrate upon the 'formal'. Hence has arisen the conception of a science of 'pure form', a science concerned with the different forms the object of thought may assume, and with the relations of implication, consistency, and so forth, between objects of different form. Since, moreover, statements concerning the relations of forms are known intuitively, a science of forms would assume the same general structure as is exhibited in the various branches of mathematics. To understand the formalization of logic, we must examine this structure in some little detail.

Any such science may be described as a 'deductive system'. It is composed of propositions; and these propositions will involve certain concepts. Among the propositions we find some which are not demonstrated in the system and others which are. The former are called 'primitive' propositions and the latter are called 'derivative'. Similarly, among the concepts there will be some which are undefined in the system, and others which are defined in terms of those which are undefined. The former are called primitive concepts and the latter derivative concepts. It is clear that any deductive system must contain both primitive propositions and derivative propositions, and both primitive concepts and derivative concepts. This follows from the nature of deduction and definition. We cannot demonstrate any proposition except from other propositions, and we cannot define any concept except in terms of other concepts. Proof and definition must begin somewhere. The propositions from which proof begins will be the primitive propositions, and the terms at which definition begins will be the primitive concepts. We must also note that in every deductive system there will be a distinction between propositions which are part of the system itself and propositions which are used in the deduction of other propositions in the system. The former constitute the distinctive subject-matter of the system. The latter (which are always logical propositions) are common to all deductive systems. It follows that if the principles of logic themselves constitute a deductive system, logic will be in the peculiar position of having the same propositions as part of the subject-matter of the system and as used by the system.

From the conception of a deductive system it is possible to elicit certain general criteria which must be satisfied by any proposition which claims to be part of the system. These may be separately stated for derivative and primitive propositions.

Every derivative proposition must be strictly deducible from the primitive propositions of the system. It is entailed by this that it must contain only primitive concepts of the system, or concepts which can be completely defined in terms of the primitive concepts of the system.

The primitive propositions must satisfy three conditions. They must be

(1) Mutually independent,
(2) Necessary to the system,
(3) Sufficient for the system.

To this it is sometimes added that they must be true, self-evident, or necessary, or that they must conform to certain empirical conditions. Certain other requirements relating to 'economy' and formal elegance are sometimes stated with reference either to the primitive propositions or to the system as a whole.

It is clear that one or other of the seven relations of opposition [1] must hold between each pair of primitive propositions. If, however, any such pair were inconsistent, the falsity of one could be deduced from the truth of the other; and we should have contradictory propositions as members of the same system. The conception of a 'system' excludes this possibility. Again, if the truth of one 'primitive' proposition could be deduced from the truth of others, it would cease, by definition, to be a primitive proposition and would rank as derivative. *Independence* remains as the only possible relation between primitive propositions.

Each primitive proposition must be *necessary*, i.e. it must be required for the deduction of some derivative proposition. This also follows from the definition.

The primitive propositions must be collectively *sufficient* to enable us to deduce all the derivative propositions of the system.

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[1] Cf *supra*, Ch. VI.

But if we define the system as the set of propositions deducible from certain primitive propositions, this criterion is also tautological. There are, however, certain considerations from which this, and the other criteria, derive their significance.

A system is not always given merely by its primitive propositions. It is sometimes given as a body of propositions independently conceived and seen to be more or less systematic. It is presented *for analysis.* For example, it is evident that prior to any systematic process of deduction, we may be cognizant of a large body of geometrical propositions. With regard to this body of propositions we may inquire: From what set of primitive propositions can the given propositions be deduced, and what further propositions belong to the same system? This opens the two lines of inquiry previously described as investigations in the backward and in the forward direction.[1] We are here concerned with the inquiry backward, from derivative to primitive propositions. The criteria of necessity and sufficiency are relative to the fact that it is *a given* body of propositions with which we are concerned. The primitive propositions must be necessary, and sufficient, for these.

The same consideration explains why it is sometimes said that the primitive propositions must be true or self-evident. Where the propositions given for systematization are themselves true the primitive propositions must also be true. If the given geometrical propositions are propositions concerning the actual world, the primitive propositions must be consistent with the facts of the actual world. If the given propositions relate to our daily numerical calculations they must be, as Russell remarks, consistent with the fact that we have two ears and one nose. But the conception of a system is applicable to a body of propositions which are merely hypothetically entertained, and to propositions which are known not to apply to the actual world. It is possible, and of interest, to construct and compare mutually incompatible systems each of which is self-consistent. Consequently, the demand for truth must not be included among the requirements of a system as such.

Similar considerations apply to 'self-evidence'. The demand for self-evidence arises from epistemological preoccupations, and from the desire to systematize our knowledge of the actual world. It arises, too, from the desire to employ the material of a system for the purposes of proof. When a proposition is a subject of doubt, we may remove doubt by showing that it follows from something concerning which no doubt arises. Consequently, if all the primitive propositions of a system are beyond doubt, we can eradicate doubt from the system as a whole. Such systems will be of special interest.

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[1] Ch. II., p. 33.

There are two senses in which a proposition may be said to be 'self-evident'. In the first it is so when it is actually seen to be true merely from inspection. In this sense, the conception involves a psychological element irrelevant to the logical character of a system. Self-evidence will be present or absent according as the individual concerned possesses or lacks the requisite powers of abstraction and 'intuition'. In the second sense, a proposition is self-evident if its truth is a necessary consequence of the nature of its terms. Its evidence is contained within itself. This does not imply that the truth of the proposition is not equally necessitated by other propositions. In fact, if the primitive propositions of a deductive system are self-evident in this sense, all the propositions of the system will be equally 'self-evident', however difficult it may be to 'see' that they are true. Systems of this nature will also be of special interest, but this character cannot be incorporated in the definition of a system.

The conception of self-evidence is associated with the doctrine that the primitive propositions of a deductive system are in some sense absolute, or absolutely prior to the derivative propositions. In general, however, it will be found that alternative systems can be constructed. On one system a given proposition will be primitive and a second derivative, and on another system the latter will be primitive and the former derivative.

The conception of logic as a deductive system of the principles of deduction, whilst very prominent in contemporary expositions, is by no means new. In the traditional logic, it was generally supposed that all the special principles of inference were derivative from three fundamental 'laws of thought,' viz.:

The Law of Identity: A is A.
The Law of Contradiction: A is not both B and not B.
The Law of Excluded Middle: A is either B or not B.

Various interpretations have been placed on these laws. They have sometimes been interpreted as expressing the fundamental properties of propositions, sometimes as expressing the properties of predicates, and sometimes as expressing the properties of classes. It has been debated whether these laws are laws of thought, or laws of things, or even merely laws of symbolism. But one tendency has been to regard them as the primitive propositions necessary and sufficient for the whole theory of deduction. It is, however, easy to show that they are not sufficient, for this purpose, and easy to show that they are not entitled to any distinctive position. It is possible to formulate many other principles equally self-evident and equally capable of providing a foundation for a generalized formal logic. In the system now to be briefly examined, we do not find any of these propositions among the primitive propositions. They occur as special deductions, and in several special forms in different departments of the system.

This generalized system of Logic, into which the traditional principles are absorbed, is commonly described as 'symbolic' or 'mathematical' Logic.[1] Both descriptions are unfortunate. The former emphasizes a purely secondary feature— the employment of an artificial symbolism. The latter description is due to the fact that the system has been developed in the course of an attempt to exhibit the ultimate foundation of mathematics.

Its exponents claim that the whole of pure mathematics is a deductive system, the primitive propositions and ideas of which are purely logical. The general theory of deduction constitutes the first stage of this system. Since, however, the concepts and propositions important for mathematics are required much more generally, we may regard this stage of the system as one of pure logic. We shall not, in any case, be concerned in this chapter with its mathematical implications.

The attempt to construct a deductive system of the principles of implication necessitates a more precise analysis of propositional forms than was attempted within the traditional system. It necessitates, moreover, a more precise analysis of different types of judgment and of the ways in which things may be known. Consequently, the development of the purely logical system has entailed a parallel development in related topics of epistemology. These developments have many important applications to the traditional problems of philosophy.

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[1] The system with which we shall be concerned is that developed by Russell and Whitehead in the 'Principia Mathematica'.

The general structure of the system may be exhibited in three stages:

(1) A theory of propositions generally, which for certain technical reasons may be regarded as primarily concerned with elementary propositions.
(2) The transition to general propositions.
(3) The theory of descriptions and classes.

At each stage there is provided a system of symbolism which admits of use as a quasi-mechanical procedure of reasoning. Such a system constitutes a 'calculus'. We have thus the calculus of propositions, the calculus of 'propositional functions' (i.e. a calculus for general propositions) and a calculus of classes.[1] Historically, the calculus of propositions and the calculus of classes were developed independently, and without special regard to the question which is the more fundamental. The system under consideration definitely places the class calculus as derivative. We commence, therefore, with the calculus of propositions.

§ 2. The Calculus of Propositions.

Primitive Ideas.—The system accepts as primitive:

(1) The idea of a proposition and the distinction between asserting a proposition and entertaining it hypothetically.
(2) One or more of the fundamental functions or relations, negation, conjunction, disjunction, etc.

In the special system with which we are concerned *negation* and the relation expressed by *either . . . or* are selected as primitive. Unfortunately, in this formulation the latter is called '*disjunction*'. To avoid confusion between this sense of the word and the usage adopted elsewhere in this book we shall always write it in inverted commas. Thus '*disjunction*' and

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[1] There is also a calculus of relations which for brevity is excluded from this survey contained in the present chapter.

'*disjunctive*' will mean what elsewhere we have meant by *alternation* and *alternative*.[1]

Implication, conjunction and equivalence are defined as follows:

Implication: $p \supset q \,.=.\, \sim p \vee q$ Df.
Conjunction: $p \,.\, q \,.=.\, \sim(\sim p \vee \sim q)$ Df.
Equivalence: $p \equiv q \,.=.\, p \supset q \,.\, q \supset p$ Df.,
(i.e. $\sim p \vee q \,.\, \sim q \vee p$).

In these formulæ '$= \ldots$ Df.' constitutes a single symbol for 'equivalent by definition'. The sign $=$ has other meanings in other contexts.

It is of the first importance to notice that, on the definition of 'implication' here given, p *implies* q means simply that, as a matter of fact, *either* p *is false or* q *is true*. In this sense *Socrates is now in England* implies $2 + 2 = 4$, since it is the case that either the first proposition is false or the second is true. The sense of implication required for this system is thus radically different from that in which the term is commonly employed, viz. that in which p *implies* q means p *entails* q. The distinctive and important property of entailment is that given p alone, q can be inferred. Entailment is the converse of the relation expressed by 'following from'. In $p \supset q$, q does not follow from p, and given p alone we cannot infer q.

Primitive Propositions.—For the system under review the following primitive propositions are required:

1. The Principle of Tautology: $\vdash : p \vee p \,.\supset.\, p$
2. The Principle of Addition: $\vdash : q \,.\supset.\, p \vee q$
3. The Principle of Permutation: $\vdash : p \vee q \,.\supset.\, q \vee p$
4. The Principle of Association: $\vdash : p \vee (q \vee r) \,.\supset.\, q \vee (p \vee r)$
5. The Principle of Summation: $\vdash :.\, q \supset r \,.\supset :\, p \vee q \,.\supset.\, p \vee r$

The foregoing are purely formal principles. In addition, two non-formal primitive propositions are required which may be designated:—

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[1] The primitive ideas here selected are those of the first edition of the 'P.M.' In the second edition true disjunction, described as incompatibility (written $p|q$ and pronounced 'p stroke q') is taken as the sole primitive idea, in terms of which negation, 'disjunction' (i.e. alternation) and all other functions are defined.

The Principle of Deduction: What is implied by a true proposition is true.

The Principle of Substitution: Whatever can be asserted about *any* proposition can be asserted concerning any given proposition.

Before discussing these principles in detail certain new symbols require explanation. The letters p, q, r, . . . as before stand for elementary propositions and the symbols $\vee$ and $\supset$ for 'disjunction' and implication. The sign $\vdash$, called the 'assertion sign', means that what follows is actually asserted, not merely hypothetically entertained. Propositions without this prefix are put forward merely for consideration or occupy merely a subordinate part of an asserted proposition. The scope of an assertion sign is indicated by the dots which follow. The dot (.) or combinations of dots (such as : or : . or : :) serve a double purpose. They stand for the relation of conjunction and they operate as brackets. Dots immediately preceded or followed by $\vee$ or $\supset$ or $\equiv$ or $=$ . . . Df. or $\vdash$ serve as brackets. Elsewhere they are signs of conjunction. The general rule is that a larger group of dots indicates an outside bracket and a smaller group an inside bracket. The scope of a collection of bracketing dots extends backwards or forwards beyond any smaller number of dots to any larger number of bracketing dots. It extends over any equal number of dots which indicate merely conjunction. This rule appears complicated but is simple in operation.

Thus, in the principle of tautology, the two dots after the assertion sign show that the whole of what follows is asserted. The single dot before the implication sign brackets all that falls between this and the two initial dots; and the dot that follows the implication sign brackets all that follows this sign.

In the principle of summation two dots are required to bracket off all that is implied by $q \supset r$, viz. that *p or q implies p or r*. Consequently, three dots are required after the assertion sign in order to assert the whole proposition. It could be written:

It is asserted that {if (q implies r) then [(either p or q) implies (either p or r)]}.[1]

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[1] It will be noticed that the sign $\supset$ can be read either as 'if . . . then' or as 'implies'. The expression 'if . . . then' itself serves as a kind of bracket, and for this reason is often more convenient than 'implies'.

We may now examine these propositions individually.

The Principle of Tautology: $\vdash : p \vee p . \supset . p$,

i.e. *If either p is true or p is true then p is true.* This principle (which merely confronts us with 'Hobson's choice') is undoubtedly trivial. But no proposition can be too trivial to be important. The importance of principles of this kind lies in their uses; and this principle is one for which deductive use can be found.

The Principle of Addition: $\vdash : q . \supset . p \vee q$,

i.e. *If q is true then either p or q is true* whatever proposition p may be, and however preposterous it may be. If it is true that to-day is Wednesday, then it is true that to-day is either Tuesday or Wednesday. If it be true that bread is good to eat, then it is true that either water is poison or bread is good to eat. It is called the principle of addition because it states that if any proposition is true any *alternative* can be added without making the resultant compound proposition false. This is obviously not the case with *conjunction:* $q \supset q . p$ would be false.

The Principle of Permutation: $\vdash : p \vee q . \supset . q \vee p$,

i.e. *If either p or q is true then either q or p is true.* The principle thus states the permutative law for the addition of propositions. (The law holds also for conjunction and incompatibility, but these cases are not required among the primitive propositions.) It expresses the fact that the relation *either . . . or* is symmetrical.

The Principle of Association: $\vdash : p \vee (q \vee r) . \supset . q \vee (p \vee r)$,

i.e. *If either p is true or (q or r) is true then either q is true or (p or r) is true.* This states the associative law for logical addition of propositions. It is, of course, also true that $p \vee (q \vee r) . \supset . (p \vee q) \vee r$, but the latter proposition is of less value in deduction.

The Principle of Summation: $\vdash :. q \supset r . \supset : p \vee q . \supset . p \vee r$,

i.e. *If q implies r then p or q implies p or r.* It is called the principle of summation because it states that any alternative can be added to each of the terms of an implication without impairing the truth of the proposition. If the fact that a man is a knave implies that he is objectionable, then the fact that he is either a fool or a knave implies that he is either a fool or is objectionable.

The foregoing principles enable us to construct endless chains of implications, but they do not in themselves enable us actually to draw inferences. For inference it is not sufficient to contemplate the fact that a certain premiss implies a certain conclusion. We require to be able to drop all reference to the premisses and simply assert the conclusion. We need some principle to the effect that if the premiss is asserted the conclusion can be asserted. This is provided by the principle of Deduction: Whatever is implied by a true proposition is true, i.e. whatever is implied by a true proposition may be asserted without reference to its premiss. This principle is required whenever we pass from an implication: *if such and such premisses are true then so and so is true* to the statement: *therefore so and so is true.* It enables us to pass from: *p implies q* to: *p, therefore q.*

The second non-formal principle which is required is one that enables us when given a proposition containing a variable to substitute for that variable any of its instances. We must be able to *apply* the principle to special cases. Now p, q, etc., are variables in all the primitive propositions. We do not need to apply them in the propositional calculus to absolutely particular cases, such as the proposition that *Socrates is mortal;* but we often need to apply them by substituting a special sort of proposition. For example, if p can stand for any proposition it can stand for any negative proposition $\sim p$, or for any conjunction $q . r$. Consequently the primitive proposition $p \vee p . \supset . p$ can be applied to the special case, $\sim p \vee \sim p . \supset . \sim p$, or to the case $q . r \vee q . r . \supset . q . r$ and so on. These substitutions are effected in virtue of the principle of substitution. It is not, however, a principle that defines *what* substitutions there are to be made.

Derivative Propositions.—We may proceed next to examine some of the derivative principles, and to illustrate the process whereby such principles are derived. The examples selected are those which are of general importance, or which have occupied a prominent place in the history of logical doctrine.

Reductio ad absurdum: $\vdash : p \supset {\sim} p . \supset . {\sim} p$,

i.e. *If any proposition implies its own contradictory it is false.*

Demonstration:

$$\left[\text{Taut.}\ \frac{\sim p}{p}\right] \vdash : {\sim} p \vee {\sim} p . \supset . {\sim} p \quad . \quad . \quad (1)$$

$$[(1)\ \text{Def. Imp.}] \vdash : p \supset {\sim} p . \supset . {\sim} p.$$

The symbols on the left, in square brackets, indicate how the propositions which follow are obtained. '$\text{Taut.}\ \frac{\sim p}{p}$' means that the proposition which follows is what the principle of Tautology becomes when ${\sim} p$ is substituted in place of p. The proposition so obtained is numbered (1), the first step in the process. In the next line the symbols in square brackets indicate how the second step is derived; viz. it is what (1) becomes in virtue of the definition of implication. This yields the required principle of the *reductio ad absurdum.* The whole operation thus consists in two steps; first, the substitution of ${\sim} p$ in place of p in the principle of Tautology, an operation we are entitled to perform in virtue of the principle of substitution; and second, an operation performed upon the 'disjunctive' component in (1), which we are entitled to perform in virtue of the definition of implication.

Corresponding theorems are derived by employing the same two steps in connection with the principles of Addition, Permutation, Association and Summation. In some cases an important theorem may be derived simply by substitution. Thus we obtain:

$\left[\text{Add}\ \frac{p}{q}\right] \vdash : p . \supset . p \vee p$, a proposition required for the proof of the principle of identity.

The Principle of Transposition.—$\vdash : p \supset {\sim} q . \supset . q \supset {\sim} p$,

i.e. *If the truth of p implies the falsity of q the truth of q implies the falsity of p.*

Demonstration:

$$\left[\text{Perm.}\ \frac{\sim p, \sim q}{p, q}\right] \vdash : {\sim} p \vee {\sim} q . \supset . {\sim} q \vee {\sim} p \qquad (1)$$

$$[(1)\ \text{Def. Imp.}] \vdash : p \supset {\sim} q . \supset . q \supset {\sim} p.$$

The first step consists in substituting, in the principle of Permutation, the negatives, $\sim p$ and $\sim q$, in place of p and q. The proposition so obtained becomes the principle of Transposition by replacing, on each side of the implication sign, the implication which corresponds by definition to the given 'disjunction'. This principle, it will be noticed, is one which underlies the compound mode *tollendo tollens* in the special case in which the consequent is a negative proposition. There are other principles of Transposition corresponding to the other cases of this mode.

The Commutative Principle.—

$$\vdash :.\ p\,.\supset.\,q \supset r : \supset : q\,.\supset.\,p \supset r,$$

i.e. *If r is implied by q, given p, then r is implied by p, given q.*

Demonstration :

$$\left[\text{Assoc.}\ \frac{\sim p,\ \sim q}{p,\ q}\right] \vdash :.\ \sim p \lor (\sim q \lor r)\,.\supset.\,\sim q \lor (\sim p \lor r) \qquad (1)$$

$$[(1)\ \text{Def. Imp.}] \vdash :.\ p\,.\supset.\,q \supset r : \supset : q\,.\supset.\,p \supset r.$$

Here, the associative principle is employed in the special case in which p and q are replaced by their contradictories. The second step consists in replacing all 'disjunctions' by implications in accordance with the definition. The principle so obtained will be employed in the derivation of the principle of syllogism.

The Principle of the Syllogism.—

$$\vdash :.\ q \supset r\,.\supset : p \supset q\,.\supset.\,p \supset r$$

$$\vdash :.\ p \supset q\,.\supset : q \supset r\,.\supset.\,p \supset r.$$

The first of these principles states : *If q implies r then (p implies q) implies (p implies r).* The second states, *If p implies q, then (q implies r) implies (p implies r).*

The first of these theories is obtained from the principle of summation and the definition of implication.

Demonstration (first principle) :

$$\left[\text{Summ.}\ \frac{\sim p}{p}\right] \vdash :.\ q \supset r\,.\supset : \sim p \lor q\,.\supset.\,\sim p \lor r \qquad (1)$$

$$[(1)\ \text{Def. Imp.}] \vdash :.\ q \supset r\,.\supset : p \supset q\,.\supset.\,p \supset r.$$

The demonstration of the second principle incorporates this theorem. We commence by effecting a substitution in the commutative principle.

Demonstration (second principle) :

$$\left[\text{Comm. } \frac{q \supset r,\ p \supset q,\ p \supset r}{p,\quad q,\quad r}\right] \vdash :: q \supset r \,.\, \supset : p \supset q \,.\, \supset .\, p \supset r :.$$

$$\supset :.\, p \supset q \,.\, \supset : q \supset r \,.\, \supset .\, p \supset r \quad . \qquad (1)$$

$$[\text{Syll. (first principle)}] \vdash :.\, q \supset r \,.\, \supset : p \supset q \,.\, \supset .\, p \supset r \qquad (2)$$

$$[(1)\ (2)\ \text{Principle of Deduction}] \vdash :.\, p \supset q \,.\, \supset : q \supset r \,.\, \supset .\, p \supset r.$$

In this demonstration the first line, complicated as it appears, is merely the commutative principle in which we have substituted the compound propositions $q \supset r$, $p \supset q$, and $p \supset r$ in the places of p, q and r.

The second step asserts the implying proposition of (1) which was established in the preceding proof. The third step in virtue of the principle of deduction consists in *inferring* the implied proposition of (1). The expression [(1) (2) principle of Deduction] means that, having before us (2) which is the implying proposition of (1), we are able to assert what is implied by this in accordance with the principle of Deduction.

With the aid of the principle of Syllogism the law of identity can be demonstrated in its application to propositions.

The Principle of Identity.—$\vdash .\, p \supset p$,
i.e. *Any proposition implies itself.*

Demonstration :

$$\left[\text{Syll. } \frac{p \vee p,\ p}{q,\quad r}\right] \vdash :: p \vee p \,.\, \supset .\, p : \supset :.\, p \,.\, \supset .\, p \vee p : \supset .\, p \supset p \qquad (1)$$

$$[\text{Taut.}] \vdash : p \vee p \,.\, \supset .\, p \quad . \quad . \quad . \qquad (2)$$

$$[(1)\ (2)\ \text{Principle of Deduction}] \vdash :.\, p \,.\, \supset .\, p \vee p : \supset .\, p \supset p \qquad (3)$$

$$\left[\text{Add } \frac{p}{q}\right] \vdash : p \,.\, \supset .\, p \vee p \quad . \quad . \quad . \qquad (4)$$

$$[(3)\ (4)\ \text{Principle of Deduction}] \vdash .\, p \supset p.$$

This example is instructive as exemplifying in detail some of the general features we have noticed in characterizing a deductive system. The fact that, on all ' common sense ' grounds, the principle does not require proof is irrelevant. It is also irrelevant that we might have taken this principle as a primitive proposition. The fact that, in the system under consideration, this principle does not appear as a primitive proposition necessitates either that it should be excluded altogether or that it, also, should be demonstrated from the primitive propositions. It should also be observed that, despite the fact that $p \supset p$ occurs in the premisses, the proof is not circular. The first proposition asserts that *if* certain propositions were true, $p \supset p$ *would be* true also; it does not assert that the hypothesis is, in fact, true. We require this relatively complicated chain of reasoning to enable us to *assert* $p \supset p$ without any hypothesis or conditions.

§ 8. General Propositions and 'Propositional Functions'.

In the preceding section we were concerned with logical formulæ which apply to propositions p, q, r, etc., without raising any questions concerning the internal constitution of these propositions. It is easy to see how, from atomic propositions, compound elementary propositions can be derived; and it is easy to see how these formulæ would apply when p, q, r are themselves elementary compound propositions. We have now to consider special applications of these formulæ (and certain new formulæ) which arise in connection with general propositions.

Consider the statement: *Every living human being breathes.* The proposition which this statement expresses is general, and must be in some way derived from elementary propositions. These elementary propositions would assume some such form as the following:

John is a living human being, and John breathes.
Henry is a living human being, and Henry breathes.
William is a living human being, and William breathes.

. . . .

The first step in generalization is *not* to add other individual cases, but to 'abstract' from the particular cases already considered. We may represent the results of abstracting from the individual cases by writing:

x is a living human being and x breathes.

This formula does not express an elementary proposition, neither a simple nor a compound proposition. It does not express a proposition at all. It expresses only a conjunction of predicates in abstraction from particulars. But what is here expressed is something rather different from what would be expressed by the formula :

Humanity and respiration.

The latter symbolises the two characters as they are symbolized when they occur as subjects of a proposition. It does not symbolize the form of the proposition in which they might so occur. In *x is human and x breathes*, on the other hand, ' being human ' and ' being a respiring thing ' are symbolized in a way which shows that they occur as predicates in a proposition ; and it shows the form of the proposition in which they so occur. We have merely to replace *x* by a word which stands for an individual and we obtain the statement of a proposition. A formula of this kind is called a ' propositional function '. To understand this conception it is desirable, first of all, not to ask why it is so described, or to endeavour to ascertain the relation of this to other senses of the word ' function ', but to proceed from the following definition :

> A propositional function is an *expression* containing one or more undetermined constituents and which expresses a proposition as soon as those constituents are determined. In other words, it is an expression containing a variable, *x*, and which expresses a proposition when a value is assigned to *x*.

Having obtained, by subtraction rather than by addition, the formula ' *x is human* ' we may proceed to consider the range of values which may be given to *x*. Three important possibilities arise :

(1) that whatever value be assigned to *x*, a true proposition will be expressed ;
(2) that there is no value whatever which assigned to *x* will yield a true proposition ;
(3) that there is some (at least one) value of *x* for which the formula would be true.

It is clear that we can assert that one or other of these possibilities is realized without actually assigning any specific value to *x*. We might, for example, assert :

(1) Whatever x may be '*x is material*' is true (and if materialism is a true philosophical doctrine our assertion would be true).
(2) Whatever x may be '*x is both round and square*' is false.
(3) There is some value for x such that '*x is human*' is true.

These assertions are genuine propositions, in spite of the fact that an x appears in their expression. They are 'general' propositions. It is important to notice the difference between these general propositions and certain elementary propositions which correspond with them in a certain way—propositions which in a sense would state the same matter of fact.

Suppose it were possible actually to enumerate one by one all the individual things that exist, and suppose we found that of each we could truly assert that it was material. We could then assert the immensely long compound elementary proposition: *a is material and b is material and c is material . . . and n is material, where a, b, c . . . n* would name, one by one, all the things there are. This proposition would then assert precisely the matter of fact which makes the general proposition '*for every x, x is material*' true. It is obvious, however, that the two propositions are fundamentally different. The same matter of fact is asserted in two fundamentally different ways. The one requires that we should be cognizant of the individuals individually; the other does not.

Similarly, suppose it is the case that there is only one man who is exactly one hundred and twenty years old to-day, and suppose we are acquainted with him. We can then assert the simple proposition '*So and so is one hundred and twenty years old to-day*', where 'so and so' is an individual we could name or point to. On the other hand, we might not know him but might assert, as a sheer guess, '*There is some one who is one hundred and twenty years old to-day*'. In the latter case we have asserted 'There is an x such that x is a hundred and twenty years old to-day'. This would be a true statement, and the fact that makes it true would be precisely the fact which is expressed in the corresponding simple statement. But, again, the two propositions are fundamentally different.

The formula *x is human and x breathes* is a conjunction of the two functions *x is human* and *x breathes*. It is thus obvious that propositional functions admit of combination in a manner analogous to the combination of propositions. That is to say, we may construct compound functions of propositional functions in the same way as we construct compound functions of elementary propositions. Taking the two functions *x is human* and *x breathes* we obtain conjunctive, implicative, alternative and disjunctive compound functions as follows :

x is human and *x* breathes.
If *x* is human then *x* breathes.
Either *x* is not human or *x* breathes.
It is not the case that *x* is human and that *x* does not breathe.

Otherwise stated commencing from the corresponding elementary propositions we obtain compound propositional functions by substituting in the appropriate expressions a variable wherever the name of an individual occurs.

Such compound expressions will, in general, be very much more important than the simpler expressions. We have not much occasion to assert such very general propositions as that everything is physical or that everything is human. The most common type of general proposition is that which is expressible in the form

For all x, if x is human, then x breathes.

This is the ordinary universal affirmative of the traditional categorical scheme on the purely intensional interpretation.

So far, we have been concerned with the 'propositional function' simply as an *expression*. The question naturally arises : What does it express ?

Compare the two cases : '*John breathes*' and '*x breathes*'. Assuming the first to be a simple proposition we may say that it contains two *elements* of different kinds and that it is characterized by a certain *form*. The element symbolized by '*John*' is of the sort which is called a *constituent* of the proposition. It is an individual, a particular, a thing, or more strictly the sort of thing that can be a subject but not a predicate in a proposition. The element symbolized by '*breathes*' is of the sort which is called a *component*. It is a *universal* or the sort of thing which can function as a predicate. The proposition has the subject-predicate form in the sense that it involves the mode of synthesis appropriate to the union of a constituent and a component when the latter requires only one constituent to be combined with it to constitute a proposition. Since there is only one way of *forming* a proposition from one constituent and one component the form does not require separate symbolization.

Considering next the expression '*x breathes*' we find a symbol for the component and the symbol x. The x does not symbolize any one constituent but serves the purpose of showing that the component in question requires one constituent in order to complete the proposition. We must thus say that the propositional function, '*x breathes*', symbolizes a certain component and the form of any simple proposition into which this component may be fitted. In the case in which the component is symbolized by an actual word, the x is unnecessary; since anyone who understands the word will understand the form of the proposition into which the component enters. But for logical purposes we require to discuss propositional functions without mentioning any special component. If we replace 'breathes' by a symbol which stands for any component we must indicate the propositional form by showing how many constituents are required. Thus we might write ϕx to show that the component ϕ requires one constituent, and $x\mathrm{R}y$ or $\mathrm{R}(xy)$ to show that the component R (being a two termed relation) requires two constituents.

Restricting ourselves to cases in which only one constituent is involved (or in which only one is to be replaced by a variable) the following system of notation will be the most generally useful.

The small letters which come early in the English alphabet, $a, b, c, \ldots$, may be employed for the *constituents* of propositions, i.e. to stand for individuals. Greek letters such as $\phi, \psi, \chi \ldots$ may be used for *components*. A simple proposition will then be represented by ϕa or ψb. (By a mere convention, devoid of special significance, the symbol for the component is placed first.) $\phi a, \psi b$, etc., will stand, therefore, for atomic propositions like p, q, etc., except for the fact that the former, unlike the latter, show the form of the proposition. The small English letters from the end of the alphabet such as x, y, z, are employed as variables. Thus ϕx will be a propositional function. We have seen that a propositional function, say ϕx, gives a general schema of an assemblage of propositions $\phi a, \phi b, \phi c, \ldots$, corresponding to all the values that may be assigned to x. We require to express assertions concerning the whole or some part of this assemblage.

The assertion that ϕx is true for all values of x is symbolized

$$(x) \,.\, \phi x.$$

This may be read, shortly, '*for all x, ϕx*'.

The assertion that there is some x for which ϕx is true is symbolized

$$(\exists x) . \phi x.$$

This is read, '*For some x, ϕx*'.

We have seen further that such assertions can be combined in the same way as elementary propositions. Thus we might assert

$$\vdash : (x) . \phi x . \supset . (\exists x) . \phi x,$$

i.e. *If it is the case that for all values of x ϕx is true, then it is the case that there is some value for x for which ϕx is true.*

Similarly, given two functions ϕx and ψx we obtain such forms as

$(x) . \phi x \supset \psi x$ if anything is ϕ it is ψ
$(x) . \phi x \vee \psi x$ of anything it is true that it is either ϕ or ψ.
$(\exists x) . \phi x \supset \psi x$ there is something such that if it is ϕ it is also ψ.

In this symbolism the traditional categorical forms become

S*a*P . . . $(x) . \phi x \supset \psi x$, i.e. For all x, ϕx implies ψx.
S*e*P . . . $(x) . \phi x \supset \sim \psi x$, i.e. For all x, ϕx implies not-ψx.
S*i*P . . . $(\exists x) . \phi x . \psi x$, i.e. There is an x which is both ϕ and ψ.
S*o*P . . . $(\exists x) . \phi x . \sim \psi x$, i.e. There is an x which is ϕ but not ψ.

An alternative notation for $(x) . \phi x \supset \psi x$ is

$$\phi x \supset_x \psi x.$$

Similarly,

$$\phi x \equiv_x \psi x . = . (x) . \phi x \equiv \psi x \quad \text{Df.}$$

It will thus be observed that $(x) . \phi x \supset \psi x$ and $(\exists x) . \phi x . \sim \psi x$ are contradictories. Similarly, with $(x) . \phi x \supset \sim \psi x$ and $(\exists x) . \phi x . \psi x$. This can be seen also in the simpler case of $(x) . \phi x$. The contradictory may be expressed $\sim \{(x) . \phi x\}$, i.e. it is not the case that for all x, x is ϕ. This is clearly equivalent to $(\exists x) . \sim \phi x$, i.e. there is at least one x which is not ϕ. Applying this to the compound function $\phi x \supset \psi x$, the universal proposition is $(x) . \phi x \supset \psi x$, i.e. whatever x may be ϕx implies ϕx. The contradictory is $\sim \{(x) . \phi x \supset \psi x\}$, i.e. it is not the case that whatever x may be ϕx implies ψx. This in turn is equivalent to $(\exists x) . \phi x . \sim \psi x$, i.e. there is at least one x which is ϕ but not ψ. The relations between E and I may be similarly stated.

It must, however, be noted that on this interpretation the universal propositions A and E do not imply the corresponding particulars, I and O. We may state

$$\vdash : (x) . \phi x . \supset . (\exists x) . \phi x,$$

i.e. if every x is ϕ then there is some x that is ϕ. Similarly,

$$\vdash : (x) . \phi x \supset \psi x . \supset . (\exists x) . \phi x \supset \psi x,$$

i.e. if every x is ψ, if it is ϕ, then there is an x which is ψ if it is ϕ. But given $(x) . \phi x \supset \psi x$ we cannot infer $(\exists x) . \phi x . \psi x$, i.e. we cannot infer that there is an x which is both ϕ and ψ; which is what the I proposition asserts.

The systematic development of the calculus of propositional functions consists partly in the formulation of principles distinctive of general propositions, and partly in exhibiting for the case of general propositions the principles which apply to all propositions regardless of their form. Among the former we find propositions such as the following:

(1) $$\vdash : (x) . \phi x . \supset . (\exists x) . \phi x,$$

i.e. if ϕx is true whatever x may be, there is some value for x for which it is true.

(2) $$\vdash :. (x) . \phi x \supset \psi x : \supset : (x) . \phi x . \supset . (x) . \psi x,$$

i.e. if ϕx always implies ψx then 'ϕx is always true' implies that 'ψx is always true'.

We obtain the analogues of the principles previously formulated in terms of p, q and r by substituting in the place of these symbols the symbols for the general proposition $(x) . \phi x$ or $(\exists x) . \phi x$. Thus the Principle of Tautology becomes

$$\vdash :. (x) . \phi x . \vee . (x) . \phi x : \supset . (x) . \phi x$$

or $$\vdash :. (\exists x) . \phi x . \vee . (\exists x) . \phi x : \supset . (\exists x) . \phi x.$$

The principle of the Syllogism becomes

$$\vdash :. (x) . \phi x \supset \psi x : (x) . \psi x \supset \chi x : \supset . (x) . \phi x \supset \chi x.$$

The principle of Identity becomes

$$\vdash : (x) . \phi x . \supset . (x) . \phi x.$$

§ 4. Descriptions and Classes.

The argument in the preceding sections has been concerned with two main types of propositions: (1) elementary propositions of the forms ϕa, $\phi a . \psi a$, etc.; (2) general propositions of the forms $(x) . \phi x$, $(\exists x) . \phi x$; and with compounds formed from these. We may next examine propositions of the forms:

A King of England was beheaded.
The present King of England is a constitutional monarch.
The Kings of England are constitutional monarchs.

The first two of these three propositions are in one sense 'singular', since they are about some one individual. The third is about the members of a class.

The two singular propositions, though singular, are nevertheless very different from propositions of the type ϕa, and from any elementary proposition combining propositions of this type. Epistemically they are similar to general propositions in one important respect. In these propositions we make an assertion concerning something with which we are not acquainted. But their difference from elementary propositions is also 'constitutive'. In each case the proposition differs from an elementary proposition in its logical properties. We can significantly assert about 'a so and so' or 'the so and so' things which we cannot significantly assert about an individual with which we are acquainted, or to which we refer by a name. For one thing, we can significantly deny the existence of the former; we cannot significantly deny the existence of the latter. *The King of France is tall* can be significantly denied on the ground that there is no King of France. '*John is tall*,' where 'John' is a genuine proper name, cannot be denied on the ground that no such person exists. '*John does not exist*' is an expression devoid of significance. The contradictory of '*John is tall*' is simply '*John is not tall*'. The contradictory of '*The King of France is tall*' is '*Either there is no King of France or There is a King of France and he is not tall*'.

Phrases of the form '*A so and so*' and '*The so and so*' as used in the examples given are called 'descriptions', the first 'indefinite descriptions' and the second 'definite descriptions'. We may call the statements in which these expressions occur 'descriptive statements'. Descriptive statements agree with general statements in expressing propositions which do not contain the individuals they are 'about' as constituents. They must be analysed in the same kind of way as general propositions —which in fact they contain.

When we speak of '*a so and so*' or of '*the so and so*' the words '*a*' and '*the*' can be significantly employed only in conjunction with a word which stands for a *component* of the proposition, i.e. something which can be symbolized by a propositional function ϕx.[1] When, therefore, we predicate some character of '*a so and so*' at least two characters are involved, one by reference to which something is described, the other which is predicated.

When we assert that 'a King of England was beheaded' we are asserting that there is something which is both ϕ (being a King of England) and ψ (having been beheaded). This is equivalent to saying that '*the joint assertion ϕx and ψx is not always false*'. In this way it is seen that the proposition is essentially akin to the general propositions $(\exists x) . \phi x . \psi x$. It differs from the latter in the same way as '*x is a man*' differs from '*x is human*'. The former is about x as a member of a class; the latter is about x as characterized by a property.

In this respect propositions about 'a so and so', propositions about 'the so and so' (singular) and propositions about 'the so and so' (plural) are alike. Provisionally, we may say that a proposition about '*a so and so*' is about *some* member of a given class; a proposition about '*the so and so*' (singular) is about a member of a class, where it is further implied that it is the sole member of its class; whilst a proposition about '*the so and so*' (plural) is about all the members of the class, where the class may contain more than one member. The two latter cases, however, require further discussion.

Definite descriptions differ from indefinite descriptions primarily in their implication of uniqueness. If it is said that '*a son of x has secured a scholarship*' the proposition declares that one out of possibly several sons of x has secured a scholarship. If it is said that '*the son of x has secured a scholarship*' there is an implication that x has only one son. The proposition is not true if x has more than one son. Apart from this, the analysis of propositions involving definite descriptions is similar to that of propositions involving indefinite descriptions. The expression '*The so and so*', like '*A so and so*', is not a name.

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[1] This is true even when we speak of 'a Bourbon' or 'The Campbells' or 'a Paul Pry'. Here the apparent proper names are abbreviated descriptions: 'A member of the family of Bourbon', 'The members of the Campbell Clan', 'A person having the character of Paul Pry'.

This can be easily seen in the case where 'the so and so' does not correspond with anything that exists. '*The King of France*' cannot be a name since there is nothing which can be properly referred to in this way. (We can, of course, name anything we like 'The King of France', but whatever we so name does not thereby become monarch of France.) There are also cogent reasons why '*The so and so*' cannot be a name even when it is applied to something which *can* be named. The example in the 'Principia Mathematica' is: '*Scott was the author of "Waverley"*'. Here we are asserting an identity between some one named and some one described, not an identity of some one named in two ways. If both '*Scott*' and '*the author of Waverley*' were names our proposition would be of the form '*Scott is Scott*', or '*Scott is Sir Walter*'. Clearly, we are not asserting anything so trivial. If both were names the proposition would be true if somebody *named* Scott '*The Author of Waverley*', even if Scott had not written 'Waverley'. But if Scott actually wrote 'Waverley' he would *be* The Author of 'Waverley' even if no one so named him.

If a definite description is not a name, a statement in which it occurs must be interpreted in the sort of way we interpret statements involving indefinite descriptions.

On the analysis offered by Russell, the proposition '*The present King of England is a constitutional monarch*' conjunctively combines three assertions:

(1) There is at least one individual who is King of England now.
(2) There is at most one individual who is King of England now.
(3) If any individual is King of England now he is a constitutional monarch.

This may also be stated in the form:

(1) 'x is a King of England now' is sometimes true.
(2) 'If x is a King of England now and y is a King of England now, x and y are identical' is always true.
(3) 'If x is a King of England now, x is a constitutional monarch' is always true.

An alternative (and still equivalent statement) is

There is a term c such that 'x is a king of England now' is true when x is c and false when x is not c, and 'if x is a King of England now x is a constitutional monarch' is always true.

This is symbolically expressed :

$$(\exists c) : \phi x \,.\, \equiv_x .\, x = c : \psi c,$$

i.e. *There is something, c, such that ϕx is true when x is c but not otherwise, and c has the property ψ.* The sign $=$ is here the sign for 'is identical with'.

When there is some one thing which satisfies the function ϕx we may symbolize the assertion that it exists by

$$E\,!\,(\iota x)(\phi x), \quad \text{That is to say}$$
$$E\,!\,(\iota x)(\phi x) \,.\, = : (\exists c) : \phi x \,.\, \equiv_x .\, x = c \quad \text{Df.}$$

Since two dots (:) appear after the first equality sign, all that follows is said to be equivalent by definition to all that precedes. (This $=$ sign, of course, here belongs to the complex symbol $\ldots = \ldots$ Df.) The symbol $(\iota x)(\phi x)$ is read '*the term satisfying ϕx*'. The whole expression $E\,!\,(\iota x)(\phi x)$ is read '*The term satisfying ϕx exists*'. Since, moreover, this is defined in terms of expressions previously introduced in connection with general propositions, all the logical formulæ embodying this notion can be introduced into the deductive system without the addition of further primitive concepts.

The analysis of assertions involving definite descriptions is important not only for purely logical purposes. It enables us to avoid certain philosophical difficulties which are apt to arise in connection with propositions asserting existence.

It has been very widely held that every statement must have a subject, and that the subject must always have denotation. On this view, some terms have both denotation and connotation, some have denotation alone; but there is no term without denotation. Consequently, when we assert that '*Griffins are so and so*' the term '*griffins*' must have denotation. This would seem to imply that in some sense there are such things as griffins. And strange as this may seem, it is not difficult to see why such a view should be held. If there are no such things as griffins, then, it might be argued, there is nothing which the proposition is about. If the proposition is about anything, what else can it be about, if not griffins ? And if it is about nothing, what is the difference between asserting something about griffins and asserting something about unicorns ? For such reasons it has been held we must distinguish different kinds of being, or different universes of discourse. Griffins and unicorns do not have the mode of being characteristic of cats and dogs, they do not exist in the universe of real things. But they have a special 'mode of being', and exist in a special universe of discourse.

This is a difficult conception, because not only unicorns, but 'round squares' and all sorts of impossible things must inhabit this second universe. In this universe, too, there must be all the heroines and heroes of novels which have yet to be written. The novels are there, too, waiting to be written, and the heroes and heroines are there waiting to be written about.

It has been suggested that an escape from this view is possible if we suppose that all the statements which seem to be about these impossible or non-existent things are really elliptical statements about ideas which do really exist. Statements about Mr. Pecksniff would be statements about an idea in the mind of Dickens or in the minds of his readers. Statements about griffins are statements about ideas in some imaginative mind, and statements about round-squares are statements about certain ideas in the minds of people who think about such things.

This account of the matter is not free from difficulty. Dickens asserted that Mr. Pecksniff lived in Salisbury. If it be certain that there is something which this proposition is about, in the sense in which 'Dickens lived in London' is about Dickens, it is equally certain that what is said about it is that the subject in question lived in Salisbury. But, clearly, we cannot restate the proposition in this form: '*The idea of Mr. Pecksniff lived in Salisbury*'.

A more satisfactory escape is provided by the analysis proposed for propositions expressed in descriptive statements. On this analysis, not every term has 'denotation'. Propositions expressed by descriptive statements need have no *constituents* whatever. They have *components* corresponding to the propositional functions which these statements involve. The *elements* of the proposition 'The griffin was annoyed' are the universals expressed by *x is a griffin* (i.e. *x* has such and such properties) and *x was annoyed.*

We may, perhaps, say that properties have a different mode of existence from that of particulars; but the properties in terms of which '*being a griffin*' is defined are real in precisely the same sense as the property of *being annoyed.* We can significantly assert that these two sets of properties jointly characterize something when, in fact, they do not, when, in short, there is no individual to which they jointly belong. The proposition is false, and it is significant without implying the existence of 'a griffin'.

Classes.—It is obvious that there is some fundamental similarity between the propositions expressed respectively by the words

The King of England is a constitutional monarch.
The Kings of England are constitutional monarchs.

The first, we have seen, can be symbolized by a formula which reads '*The term satisfying the function* ϕx *is so and so*'. The second can similarly be symbolized by a formula which reads '*The terms satisfying the function* ϕx *are so and so*'. In other words, the first asserts there is an individual (with whom we are not necessarily acquainted) such that he has the property ϕ and he also has the property ψ. The second asserts that there are individuals which have ϕ and they all have also the property ψ. Now these individuals constitute a class, and the proposition is thus in a sense about a class. In the analysis of propositions expressed by descriptive statements employing 'the' in the plural we require to examine the nature of a class.

It has been seen in § 3 of this chapter that from propositions of the form

John is human and John breathes,
Henry is human and Henry breathes,

i.e. from propositions of the form $\phi a \,.\, \psi a$, $\phi b \,.\, \psi b$, etc., we may by abstraction come to think of a certain relation between the two properties ϕ and ψ. We cannot think of the properties ϕ and ψ except as things that characterize, and this relation holds only in virtue of the fact that they may characterize the same thing; but we can think of this relation without thinking of any particular thing which they jointly characterize.

Similarly, we can assert the fact expressed by

$$(x) \,.\, \phi x \,.\, \equiv \,.\, \psi x$$

without calling to mind all the things each of which are jointly characterized by ϕ and ψ.

We may now observe that in some way we can think of all these things (without knowing any of them individually) in such a way as to enable us to assert that all the things which have ϕ *are identical with* the things which have ψ.

Here it is to be noted, our assertion is not directly to the effect that a certain relation holds between two *properties*, for ϕ and ψ are not identical. The identity holds between the *things* which have these properties.

It is, perhaps, a strange and remarkable fact that we can pass judgments about whole sets of things when we are not acquainted with any members of these sets; but the fact itself is indubitable. It is presumably connected with the fact that we can think of properties like ϕ and ψ without thinking of the individuals characterized, and with the fact that a property is the sort of thing that must characterize.[1]

The transition from the judgment: *Whatever* x *may be*, ϕ *is equivalent to* ψ to the judgment: *the terms which satisfy* ϕx *are identical with those which satisfy* ψx is the transition of thought which previously has been referred to as that from the intensional to the extensional 'interpretation' of a proposition. The suggestion that the latter is in some way derivative from the former constitutes one of the reasons for asserting that the intensional reading is the more fundamental.[2] In passing from a general proposition of the form: *For all values of* x, ϕx *is equivalent to* ψx to one to the effect that '*The things which are* ϕ *are identical with the things that are* ψ' we are passing from intension to extension, from properties to classes.

A class term, like a description, is not a name. As we have seen, we can make significant assertions about a class without being acquainted with any of its members. In consequence a class cannot be identified with its members. This is clearly seen in the case of what is called the 'null class', the class which has no members. This is the class to which unicorns belong. Nor can a class term be identified with the propositional function by which it is defined. We can in fact specify the members of a class without reference to any property whatever. Pointing one by one to a series of individual things we might assert that '*This and that and that and that are mine*'.[3] Similarly, we can designate a class of men by mentioning each member by his proper name. It is clear, however, that this procedure is less generally available than that of referring to a class by a defining property. It cannot be employed in the case of the null class nor of any class containing more than a small finite number of members. In general, therefore, we are concerned with a class as comprising the things which have a certain property. In the terminology of the 'Principia Mathematica' these things are the 'terms satisfying a certain propositional function'. For this expression 'the terms satisfying ϕx' we may write, shortly,

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[1] This is not to say that every property, simple or complex, which we think of must belong to something. There are further complications here into which we need not enter.

[2] Ct. *supra*, Ch. V.

[3] Assuming that 'this' and 'that' are genuine proper names—which as ordinarily employed they almost certainly are not.

$$\hat{x}\,(\phi x).$$

Thus, $\hat{x}(\phi x)$ and the more commonly employed symbol $\hat{z}(\phi z)$ are symbols for classes. But we have seen that the same class may be defined by two or more different functions. The terms which satisfy the function '*x is an equilateral triangle*' are identical with the terms that satisfy the function '*x is an equiangular triangle*'. Hence we may assert generally

$$\vdash :. \psi x \,.\, \equiv_x . \,\chi x : \supset . \,\hat{z}(\psi z) = \hat{z}(\chi z),$$

i.e. If ψx is equivalent to χx for all values of x then the terms which satisfy one function are identical with those which satisfy the other, and vice versa. In other words, two classes are identical when their defining functions are formally equivalent. Similarly, two classes are identical when they have the same members. The relation of an individual to a class of which it is a member is symbolized by the Greek letter epsilon, and the relation is called 'the epsilon relation'. Thus the statement that x is a member of the class defined by ψ may be written

$$x \epsilon \hat{z}(\psi z).$$

Since $\hat{z}(\psi z)$ is determined by the property ψ we may state

$$x \epsilon \hat{z}(\psi z) \equiv \psi x,$$

i.e. *x is a member of the class determined by ψ when and only when ψx is true.*

Consequently the statement that two classes are identical when and only when they have the same members may be written

$$\vdash :. \hat{z}(\psi z) = \hat{z}(\chi z) \,.\, \equiv : x \epsilon \hat{z}(\psi z) \,.\, \equiv_x . \,x \epsilon \hat{z}(\chi z).$$

It is desirable to employ a shorter symbol for a class like $\hat{z}(\psi z)$ and one that is independent of the specific function by which the class is defined. For this purpose small Greek letters, α, β, γ . . . are employed—excepting of course letters such as ϵ, ϕ, ψ, χ, etc., which are required for other purposes. Thus the proposition just stated may be restated more simply:

$$\vdash :. \alpha = \beta . \equiv : x \epsilon \alpha . \equiv_x . x \epsilon \beta .$$

The calculus of classes follows immediately from the foregoing considerations. It arises from the transition from assertions concerning properties to assertions concerning the things to which these properties belong. In this account of the matter three points are of special importance. The first is that on this view the calculus of classes is derivative from the calculus of propositions. This reverses the historical order, and in a sense the psychological order, of exposition. In the history of symbolic logic the calculus of classes has been developed independently of the calculus of propositions; and since classes have been supposed to be the constituents of which propositions are composed, it has been held that they are more fundamental than propositions. This tendency has been reinforced by the observation that the calculus of classes is perhaps psychologically more 'obvious'. The unanalysed concept of a class is exceedingly familiar, and in virtue of the analogy between a class and a closed figure the relations between classes are easily brought to mind. It is chiefly the difficulty of defining the ontological status of a class which compels us to derive propositions concerning their relations from propositions concerning propositions themselves.

The second point to notice in the foregoing account is that it enables the calculus of classes to be constructed without the introduction of the concept of a class as a new primitive idea, and without assuming that classes are a part of the 'ultimate furniture of the world'—without assuming, in short, that classes really exist. The theory of the *Principia Mathematica* provides a notation to represent classes, but avoids the assumption that there are really any such things. To this point we shall return after a brief examination of the calculus itself.

The propositions of the class calculus are to a large extent analogous to those of the system of propositions. They result from the substitution of classes α, β, γ, . . . in place of the propositions p, q, r, . . . and from corresponding substitutions for negation, 'disjunction', implication and the other relations in terms of which the compound functions are constructed.

Corresponding to the negative function, $\sim p$ or $\bar{p}$, of the proposition p there is the negative of the class α which is written $-\alpha$ or $\bar{\alpha}$. As the class α consists of all the things of which $x\epsilon\alpha$ may be truly and significantly asserted, so $-\alpha$ consists of all the things of which $x\epsilon\alpha$ may be significantly and truly denied. If the class α is the class of *men* the class $-\alpha$ consists of all those things of which we may assert that they are not men. Thus

$$-\alpha \,.=.\, \hat{x}(x \sim \epsilon\alpha) \quad \text{Df.}$$

Given two classes α and β there is also a class which consists of the things which are contained either in α or in β. This is called the logical sum of α and β and is symbolized by $\alpha + \beta$. Thus, given the class of *sons* and the class of *daughters*, we obtain by their sum the class *children*. (Of course, there is not always in common language a distinctive class-name for the sum of two classes which have distinctive class-names.) Defined in the manner in which negation was defined

$$\alpha + \beta. =. \hat{x}\,(x\epsilon\alpha \vee x\epsilon\beta) \quad \text{Df.}$$

The logical sum of two classes is the analogue of the alternation of two propositions $p \vee q$. The latter is therefore sometimes described as the logical sum of the propositions p and q.

So, too, given two classes α and β there is the class of things contained in both. This is symbolized by $\alpha \times \beta$, or simply by the conjunction $\alpha\beta$, and is called the logical product of the two classes. In this way, from the two classes, the deaf and the dumb, we obtain the class of deaf-mutes, viz. those who are both deaf and dumb. Technically defined

$$\alpha \times \beta \,.=.\, \hat{x}(x\epsilon\alpha \,.\, x\epsilon\beta) \quad \text{Df.}$$

This is obviously analogous to the conjunctive function of two propositions, so the latter can be described as the logical product of the two propositions.

Corresponding to the relation of *implication* we have in the calculus of classes the relation of *inclusion*. If $x\epsilon\alpha$ implies $x\epsilon\beta$

the class α is included in the class β. The relation of inclusion is symbolized by a reversed implication sign

$$\alpha \subset \beta \,.=.\, (x) x \epsilon \alpha \supset x \epsilon \beta \quad \text{Df.}^{1}$$

Two further conceptions are of special importance in the calculus of classes, 'the Universe' or 'Universal class' and the 'Null class'. 'The Universe' is the class of all individuals. It is commonly symbolized by 1. All other classes are obtained by selecting individuals out of the universal class. If we select from all individual things those that are material we obtain the class of material things. The things that are left are the class of immaterial things. It is thus obvious that the process whereby we obtain the negation of a class is analogous to subtraction: $-\alpha = 1 - \alpha$ (where the second $-$ has the meaning of subtraction). Obviously, too, $\alpha + -\alpha = 1$. The null class is the class which contains no members. It is required in the calculus of classes for the class determined by propositional functions expressing properties which belong to nothing; e.g. the property of being a round square. It is symbolized by 0.

The relation between the universal class, the null class and any other class α can be summed up as follows:

$1 \subset 1$ i.e. the universe is included in the universe.
$\alpha \subset 1$ i.e. any class is included in the universe.
$0 \subset 0$ i.e. the null class is included in the null class.
$0 \subset \alpha$ i.e. the null class is included in every class.

In the systematic development of the calculus of classes we obtain all the general principles of traditional logic on its extensional interpretation, and a vast body of other principles as well. The following may be selected for illustration. In each case the principle will be recognized as analogous to others previously formulated for propositions.

The Principle of Tautology.

For the addition of classes $\alpha + \alpha = \alpha$
For multiplication $\alpha\alpha = \alpha$

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[1] Notice, (1) that this is the only commonly recognized use of the sign $\subset$ as opposed to the special use to which it has been put in previous chapters for the relation of counter-implication; and (2) that $\alpha \subset \beta$ is not itself a class, so that its definition is of a form which differs from those employed for negation, addition and multiplication.

These principles indicate one of the logical properties of addition and multiplication—one of the properties in virtue of which the 'algebra of logic' differs from ordinary algebra. It is obvious that the class consisting of all who are either poets or poets is simply the class of poets; and the same applies to the case in which we first of all select from the universe all who are poets, and then from this class again select those who are poets.

The Principle of Association.

For Addition $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
For Multiplication $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

The first form of the principle asserts that if, for example, we add to the sum of poets and musicians the philosophers, we obtain the same class as if we had added to the poets the sum of the musicians and philosophers. The second form asserts that if we select from the class composed of men who are both tall and handsome those who are in addition clever, we obtain the same result as if we had started from the total class of tall men and selected those who are also both handsome and clever.

The Principle of Commutation.

For Addition $\alpha + \beta = \beta + \alpha$.
For Multiplication $\alpha\beta = \beta\alpha$.

This principle exhibits the symmetry of the relations of addition and multiplication. The corresponding principle for propositions exhibits the symmetry of the relations of 'disjunction' and conjunction.

The Principle of Transposition.

$$\alpha \subset \beta \,.\, = \,.\, -\beta \subset -\alpha,$$

i.e. *If α is contained in β then not-β is contained in not-α*; or, *If all S is P then all non-P is non-S.* The principle is that which underlies contraposition, and is the class analogue of the *tollendo tollens* of the 'compound modes'.

The Principle of the Syllogism.

$$\alpha \subset \beta \,.\, \beta \subset \gamma \,.\, \supset .\, \alpha \subset \gamma.$$

This is the syllogism in Barbara. In view of the fact that the epsilon relation is to be distinguished from the relation of inclusion we require also the form

$$\alpha \subset \beta \,.\, x \epsilon \alpha \,.\, \supset .\, x \epsilon \beta.$$

This provides for the case in which the minor term is 'singular'. For example, '*All men are mortal*' and '*Socrates is a man*', implies '*Socrates is mortal*'.

§ 5. Incomplete Symbols and Logical Constructions.

It has been noticed that the theory of classes adopted by the logical system with which we have been concerned—although it provides a notation to represent them—avoids the assumption that there are such things as classes. If classes are genuine constituents of the world, that symbolism provides a simple representation of their properties. On the other hand, if there are no such things, all the propositions of the class calculus can be interpreted in such a way that they remain true and important.

The doctrine that classes are not part of the ultimate furniture of the world has also been expressed by saying that they are 'fictions', 'symbolic fictions', 'logical constructions' or 'incomplete symbols'. This doctrine has, further, been applied to 'descriptions'. It is, moreover, a doctrine of fundamental importance not only in logical theory but in many discussions of the problems of philosophy. Not only are such logical entities as classes described as fictions or constructions, but also material objects, 'selves', and a large number of other things concerning the nature of which philosophy endeavours to give account. The general theory of constructions is still in many ways obscure and incomplete. It is important, however, to indicate some of the broad issues involved.

The first step is to distinguish 'incomplete symbols', and other expressions referring primarily to symbols, from 'logical constructions' and other expressions which appear to refer more directly to what is symbolized. We may assert (1) that 'X' is an incomplete symbol, or (2) that X is a logical fiction. The two assertions are logically connected. The first statement is *more directly* about some symbol in association with other symbols—about ' X ' as it occurs in a larger expression. It is, however, *less directly*, about what is symbolized, i.e. about the relation of ' X ' in the larger expression to whatever is so expressed. The second statement is *more directly* about what is expressed by the total statement, but *less directly* it is about the relation of what is so expressed to the symbols in which it is expressed. More particularly, it is about the relation of what is expressed to some part of the set of symbols employed.[1]

We thus find that the facts expressed by the two statements ' " X " *is an incomplete symbol* ' and by ' X *is a logical fiction* ' are related as *ArB* is related to *BrA*, where A is some symbolic expression and B something which may be symbolized. It will be convenient to examine this connection from each of its two ' ends '.

The most common case is that in which the ' grammatical subject ' of a statement is an incomplete symbol. Thus, in the assertion ' *The class of men contains Socrates* ' the phrase ' *the class of men* ' would be an incomplete symbol. Similarly, in the statement ' *The King of France does not exist* ' the phrase ' *the King of France* ' is an incomplete symbol. In the notation previously described $(\iota x)(\phi x)$, in $\sim E\,!\,(\iota x)\,\phi x$, is an incomplete symbol.

The most obvious implication of this statement is negative: viz. that the grammatical subject of the statement does not stand in the same relation to the logical subject as does any grammatical subject which ' directly represents ' the logical subject. The grammatical subject and the grammatical predicate of a sentence may be defined by their syntactical properties. A subject is a word which can enter into certain positions in a sentence or which undergoes certain variations; whereas a grammatical predicate occurs in other positions and undergoes other variations. The logical subject and the logical predicate may be defined according to the manner in which they enter into the judgment. In the simplest types of sentence, e.g. when the subject is a proper name, and the predicate refers to a quality, we may say that the grammatical subject directly represents the logical subject and the grammatical predicate directly represents the logical predicate.[1] When the grammatical subject is an incomplete symbol this relation does not hold. For simplicity, we may say that a grammatical subject, when it is not an incomplete symbol, directly represents a *constituent* of the proposition with which the thinker is acquainted. A grammatical predicate, in a corresponding way, directly represents a *component* of the proposition. Now it is obvious that in stating that '*The King of France is a constitutional monarch*' these relations do not hold. There is no constituent of the proposition which can stand in the required relation to the grammatical subject. There is no King of France, in fact; and there is no reason to suppose that any other individual takes his place as a constituent of the proposition. Even when the description does in fact describe the same general considerations hold. The statement '*The King of England is a constitutional monarch*' is of the same grammatical form. There is, in fact, an individual who may be so described; but he does not enter as a constituent into the proposition. The proposition can be asserted by some one who is not acquainted with him. Since the judgment, the proposition and the sentence are of the same type in both cases, we must say that if 'the King of France' is an incomplete symbol so is 'the King of England', when similarly used. The two statements differ in their relations to fact, but this does not affect the form of the *proposition*. It is, in general, to be desired that an incomplete symbol should be defined, not in terms of its relation to any fact but in terms of its relation to the judgment expressed.[2]

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[1] We may say that in the first case the symbol ' X ' is what the assertion is about, but that in the second case ' X ' is merely *used* to say something about something else. The complication arises that it is used to say something indirectly about ' X '.

Negatively, therefore, we may say that 'X' is an incomplete symbol if 'X' does not directly stand for any *constituent* of the judgment in whose verbal expression it occurs. From this certain important consequences follow. When we replace the statement in which 'X' occurs by one in which every grammatical element directly represents an element in the proposition, the symbol 'X' will disappear. The substituted statement will show the form of the proposition in a way in which the original statement did not. We may next consider the problem in relation to the 'construction'.

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[1] The relation is only relatively direct. It is not a simple diadic relation.

[2] We might have said in terms of its relations to the proposition, but for the fact that 'propositions' are themselves perhaps logical 'constructions'.

The conception of a 'logical fiction' or 'logical construction' may be illustrated in the special case of the type of entity to which we appear to ascribe the attributes of personality when it is quite obvious that no such entity exists (in the sense in which individual persons seem to exist). We say, for example, that France distrusts Germany, that England was afraid of Napoleon and so on. In all such cases we seem to be describing some entity which combines the properties of a nation with those which characterize an individual mind or consciousness. It seems obvious that in some way these statements require to be translated into terms in which no such entity would be referred to.

Let us consider the simpler case of a special association. A certain society—let us call it the Society for the Promotion of Good Causes—exhibits its foresight and sound judgment by inviting Mr. A. to become its president. Clearly the S.P.G.C., which displays such characteristics and *pursues ends* appropriate to its title, cannot be regarded as belonging to the ultimate furniture of the world. All that we can wish to say in any statement in which its title occurs must be capable of being said in a different way, and in terms of things that really exist. Nevertheless, we cannot directly substitute for statements about the S.P.G.C. precisely the *same* statements about its members. *The Society decided to invite Mr. A. to become its president.* This might well be true even if not a single one of its members wished this to be done. The members elected the Executive Committee. The Committee had before it the names of three candidates for the presidency, A, B and C. Half the Committee wanted B; the other half wanted C. They decided to vote on the following plan. Each of the ten members of the Executive wrote down the three names in the order of his preference. The first name would carry three votes, the second two, and the third one. The B-party, being strongly opposed to C, wrote down the candidates in the order B, A, C. The C-party correspondingly wrote C, A, B. Under such conditions A would obtain 20 votes, B 15, and C 15; and A would be elected.

Though no one wanted A, it can be truly said that this decision expressed the will of the Committee—in the sense in which a Committee can be said to have a will. A similar type of analysis might be given of all the actions, aspirations, fears and other occurrences in the intellectual and emotional life of the society.

Now in setting out in detail what happens when ' a Committee, decides to do something ' we are not substituting a true statement for a false one, we are substituting one true statement for another. But the two statements differ in important respects.

In the more detailed statement we find a grammatical subject-term which represents members of the Society, or members of the Committee. Such expressions as ' The S.P.G.C.', ' The Executive Committee ', etc., disappear. They disappear in the way in which an ' incomplete symbol ' disappears when, for a statement about a class, we substitute a statement concerning a property.

In the second place, we notice that the predicate-terms in the more detailed statements are either changed, or undergo a change in the way they are used. The sense in which a committee may be said to ' decide ' is different from that in which an individual ' decides '. This fact may be more precisely expressed by saying that when the grammatical subject is an incomplete symbol, the grammatical predicate is subject to ' systematic ambiguity '. Now, generally, the terms which stand for *components* (as opposed to constituents) of the proposition are subject to ambiguity of this kind. In the two statements : ' *England feared Napoleon* ' and ' *Josephine feared Napoleon* ', where ' Josephine ' and ' Napoleon ' are true proper names, it is only the word ' fears ' which is subject to systematic ambiguity. Ambiguity is systematic when the meaning of a term is functionally dependent upon its context. A word representing a component, when employed in conjunction with an incomplete symbol, will have a different meaning from that which it bears when employed with a proper name.

The fact that a statement involving an incomplete symbol expresses the same proposition as may be expressed in a statement not involving an incomplete symbol enables us to define the sense in which a logical construction is ' constructed out of something '. *The Committee* is constructed out of facts concerning the individuals forming the Committee. Any ' logical construction ' may be said to be ' constructed ' out of the elements of the corresponding proposition which does not involve the construction. A ' construction ' is the correlation of the elements from which it is constructed. Similarly, an ' incomplete symbol ' is the correlation of the symbols which would directly represent the elements of the proposition expressed.

Combining the two foregoing considerations, we may say that we cannot assert precisely the same thing about a logical construction and about the things out of which it is constructed. A chair is sometimes said to be a logical construction. Whether it is or not we can truly say, on suitable occasions, that '*A is sitting on a chair*'. If, however, we may truly say that a chair is a logical construction constructed out of the chain of events which constitute the history of the chair, we cannot say that *A is sitting on a chain of events.* In substituting for ' a chair ' the expression ' a chain of events ' we must make some corresponding adjustment in ' A is sitting on '. We must either substitute another phrase or change the sense of the phrase as here employed.

A case of special importance is the assertion of existence. We have seen that one of the implications of the assertion that '*so and so is a construction*' is that there is no such thing as ' a so and so ' among the items which constitute the ultimate furniture of the earth. In *one* sense, therefore, '*the so and so*' does not exist. To say this, however, is misleading. The fact is better expressed by saying that ' existence ' like ' fearing ' and ' sitting on ' in our previous examples, is subject to systematic ambiguity. The sense in which a chair exists (if a chair is a construction) is different from that in which the entities from which it is constructed exist. '*Chairs exist and are logical constructions*' is not a self-contradictory statement; but it implies that ' chairs are not part of the ultimate furniture of the world '.

Analysis is rarely carried to the point at which the ultimate elements are reached. The entities out of which a construction appears, at first sight, to be constructed often themselves turn out to be logical constructions. We provisionally assumed that a committee was constructed out of its members. But these members may be constructions. The fact that '*Mr. X decided to vote for Mr. B*' must perhaps be analysed much as we analysed '*The committee decided to elect Mr. A*'. ' X's ' decision must perhaps be analysed into a chain of inclinations and counter-inclinations, reflexions and doubts, issuing finally in the writing of certain marks on paper. More generally, Mr. X is analysed in terms of a chain of atomic psychical, physical or psycho-physical facts. Any statement in which 'Mr. X' appears must be replaced by one from which 'Mr. X' has disappeared. He is not one of the ultimate inhabitants of the world.

It is because the procedure of analysis required in the case of descriptions and classes applies, on some views, to tables and chairs and members of committees that the conception of a logical construction is of such fundamental importance in current philosophical thought.

BIBLIOGRAPHICAL NOTES.

CHAPTER IX.

The generalized logical doctrine briefly introduced in the present chapter is the product of two interconnected, but partly independent, lines of development. On the one hand, we have a series of attempts to construct a purely symbolic apparatus for reasoning; and on the other hand, various endeavours to provide a complete logical analysis of the principles of mathematics.

The conception of a universal scientific language has a long history, but in the hands of Leibnitz it assumed new importance as part of a more ambitious project. It is here but one of two connected notions—the *characteristica universalis* and the *calculus ratiocinator*. The former was to provide a universal notation for all simple concepts, the latter an apparatus by which all derivative concepts might be constructed by purely symbolic operations.

Such a calculus, however, began to assume a practicable form in the more restricted system of George Boole, whose works are generally regarded as originating the modern phase of symbolic logic (**An Investigation of the Laws of Thought**, 1854, reprinted 1916). In the subsequent development of this algebra of logic contributions were made by **W. S. Jevons** (**Pure Logic**, 1864), **J. Venn** (**Symbolic Logic**, 1894), and **C. S. Peirce** ('**Description of a Notation for the Logic of Relations**,' **Memoir of the American Academy of Arts and Sciences**, Vol. IX., 1870, and '**On the Algebra of Logic**', **American Journal of Mathematics**, Vol. III., 1880). This logic assumed its present form in the hands of E. Schroeder (**Vorlesungen über die Algebra der Logik**, 1890-1895). It is hence commonly designated the Boole-Schroeder Algebra.

The second line of development leading to the modern system of Logic is that derived from **Frege** and **Peano**, the concern of these authors being to ascertain the ultimate foundation of mathematical reasoning. The contribution of Peano is the 'arithmetizing of mathematics' that of Frege is the 'logizing of arithmetic,' the latter being historically prior to the former. (**Frege, Die Grundlagen der Arithmetik,** 1884; **Grundgesetze der Arithmetik,** 1893-1903; **G. Peano, Formulaire de Mathematique,** 1895-1908.)

This second line of development, assimilating the first, culminates in the monumental work of **A. N. Whitehead** and **B. A. W. Russell,** the **Principia Mathematica** (1st edition, 1910-13, 2nd edition, 1925-27). An earlier (and in some respects more detailed) exposition of the fundamental concepts of this work is contained in **B. A. W. Russell's Principles of Mathematics** (1903).

Important developments and modifications (some of which are incorporated in the second edition) have followed the publication of this book. In particular the student should note the contribution of Sheffer and Nicod in reducing the number of primitive propositions and concepts required by the system. **(H. M. Sheffer, 'A Set of Five Independent Postulates for Boolean Algebra', Transactions of the American Mathematical Society,** XIV., 1913; **J. Nicod, 'A Reduction of the Number of Primitive Propositions of Logic', Proc. Cambridge Philosophical Society,** XIX., 1916.)

Ludwig Wittgenstein's Tractatus Logico-Philosophicus contains some of the most important clues to the development of philosophical aspects of modern logic since the first edition of the **Principia. F. P. Ramsay's Foundations of Mathematics** should also be consulted in this connection. Important articles relevant to special topics will be found in **Mind.**

Until recently no elementary introduction to modern logic was available. The omission is supplied by **L. S. Stebbing's Modern Introduction to Logic,** and by **R. M. Eaton's General Logic** (the latter being more detailed in the exposition of the logical calculus). The same author's earlier work, **Symbolism and Truth,** contains useful discussions of the more philosophical aspects of the subject.

B. A. W. Russell, An Introduction to Mathematical Philosophy, is an essential volume for the student. Other valuable works are **L. Couturat's** little **Algebra of Logic** (Paris, 1905, English translation, 1913), **C. I. Lewis, A Survey of Symbolic Logic,** 1918, and the **Symbolic Logic** of **C. I. Lewis** and **C. H. Langford,** 1933. **The Encyclopædia of the Philosophical Sciences,** Vol. I., **Logic,** contains two valuable articles, one by J. Royce and the other by Couturat.

CHAPTER X.

FORMAL LOGIC AND SCIENTIFIC INQUIRY.

§ 1. Classification and Division.

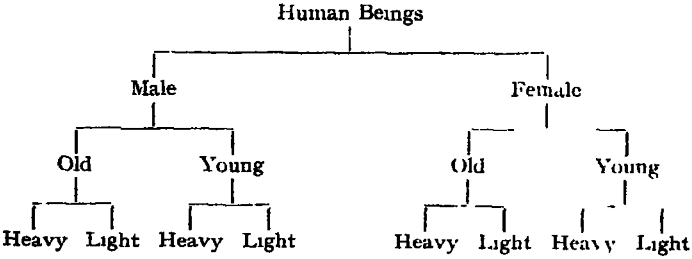
THE generalized theory of deduction outlined in the preceding chapter escapes many of the lines of attack to which the traditional system is exposed. It provides an account of the constitutive basis of inference which is both comprehensive and systematic, and the concepts it employs are analysed with some precision. On the other hand, it does not profess to provide the plain man with a guide to 'the art of thinking,' nor to supply the scientific worker with an instrument of research. Here, as elsewhere, the development of science is the fruit of specialization. The four general aims implicit in traditional logic awaken divergent trains of thought. The general conditions of efficiency in thinking may be more profitably studied as a branch of applied psychology. To this study Logic can offer at best but a slender contribution. The psychologist, of course, cannot study thought without regard to its constitutive conditions; and so far he depends upon the logician. But apart from this, even the topics of traditional logical doctrine which might at first sight appear to be most relevant to the problem are of dubious value. They consist in the main of a tabulation of the fallacies and of isolated observations on the dangers that arise from ambiguity, vagueness and ill-defined terms. On these matters there are important things to be said, but for adequate discussion they call for a thoroughgoing psychological approach.

The limitations of traditional logic in relation to the theory of scientific method are in many respects very much more serious. Nor are they remedied by the generalized theory of deduction. The limitations are in large measure the inherent limitations of the principles of deduction. Not all reasoning is deductive; and so far as this is so, the principles of deduction simply fail to apply. Opposed to deduction is the procedure of generalization, the theory of which requires an independent approach. But whilst in essence inductive, scientific method employs deductive reasoning ; it employs, if only in a subsidiary way, all the operations with which traditional formal logic has been concerned. Nevertheless, in its bearing even upon the merely incidental phases of scientific thought, traditional formal logic is of very doubtful relevance to the problems with which the scientist is concerned. This is due, in the main, to the rigid restrictions which logicians have imposed upon themselves in the study of propositional forms. Some of the most important problems come to light when assertions are considered, not merely as either ' universal ' or ' particular ', but in various ways numerically precise. Others emerge when they are considered, not merely as either matter-of-fact or ' necessary ', but also as in varying degrees problematic. The possible extensions of the study of propositional forms which these considerations suggest lead directly to the general theory of statistics, and to the theory of probable inference. For reasons which will appear as we proceed, it will be convenient to approach these topics via the traditional doctrine of classification and division.

Classification and division are sometimes described as operations subsidiary to induction. This account, though in some ways too restricted, draws attention to the important fact that a satisfactory classification must conform to two sets of conditions. It must conform to certain technical and formal conditions, and it must satisfy certain material requirements upon which its practical and scientific utility will depend.

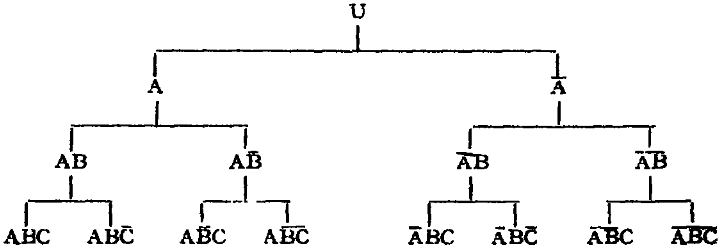
The term ' classification ' applies to the case in which we commence with *individuals*. Individuals are arranged in groups, and the groups so formed are combined into larger groups. The process ends with a single all-embracing class to which all the individuals are referred. The term ' division ' is applied to the reverse procedure. We commence with a single class. This is subdivided on the basis of differences which are observed between the different kinds of individuals it contains. The sub-classes so obtained may be further subdivided. For reasons which will shortly appear it is best to consider first the process of division.

Division commences with a single class—say the class of human beings. These we may divide into sub-classes by reference to any attribute whatever. We might ' divide ' human beings according as they are under twelve feet tall or over. The fact that the second sub-class would contain no members does not affect the formal character of the division. In general, however, division proceeds by reference to characters which are sometimes present and sometimes absent. Thus we might reasonably divide human beings according to sex. This yields two sub-classes, Males and Females. The attribute by reference to which the division is effected is called the *fundamentum divisionis* or the *principle of division.* The two classes, Males and Females, may be again divided according to a second principle of division, e.g. Age. Selecting 35 years, say, as the line of division, we could sub-divide each group into Old and Young. These again could be divided according to weight, e.g. into those above average weight, and those not above average weight. We should thus obtain a set of classes related as shown in the following table:



In the traditional technical language, the single class from which division starts, or classification ends, is called the *Summum Genus.* The intermediate classes are called *subaltern genera*, and the classes with which division ends the *infimae species.* The terms *super-ordinate, sub-ordinate* and *co-ordinate* are used to describe the relations of any class to any other class standing, respectively, below it, above it or on the same level with it. In division, as a purely formal operation of constructing possible classes and sub-classes, it is obvious that the process may be indefinitely extended. It is limited only by the number of principles of division which may theoretically be introduced. In practice, the operation ends when we reach a point at which further division would yield only classes that contain no members. In classification, on the other hand, this limit is absolute. Starting with individuals to be classified, the *infimae species* must each contain at least one member.

In the example given, the division is described as *dichotomous*, since at each step every class is sub-divided into two, and only two, sub-classes. These are mutually exclusive and jointly exhaustive of the class given for division. One sub-class is a positive class; the other is its contradictory or correlative negative class. It is a matter of indifference which is described as positive and which negative. We could describe Females as Not-males, or Males as Not-females. If we represent the classes and their contradictories by capital letters, A and $\bar{A}$, B and $\bar{B}$, C and $\bar{C}$, etc., corresponding to the attributes, *a*, *b*, *c*, etc., which serve as principles of division, the sub-classes are then represented by combinations of these symbols AB, $A\bar{B}$, $\bar{A}B$, $\overline{AB}$, ABC, $AB\bar{C}$, $A\bar{B}C$, $A\overline{BC}$, etc., as represented in the table below:



The symbol U is employed for the 'universe' or *summum genus* given for division. It may, of course, be added to each of the other symbolic groups, UA, $U\bar{A}$, UAB, $UA\bar{B}$, etc. It differs, however, from the other symbols in the table in being common to all the class symbols, i.e. in having no contradictory. Hence we may say that any symbol which is common to all symbolic groups in such a tabulation specifies the universe with which the division is concerned.[1]

Dichotomous division is, in general, somewhat artificial, and in many cases a *manifold* classification (or division) is more usefully employed. This is illustrated by the following:

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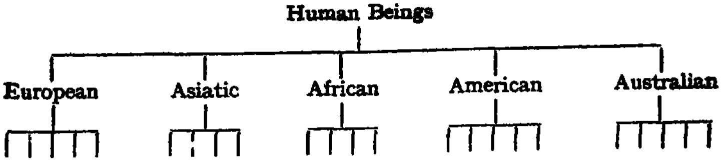
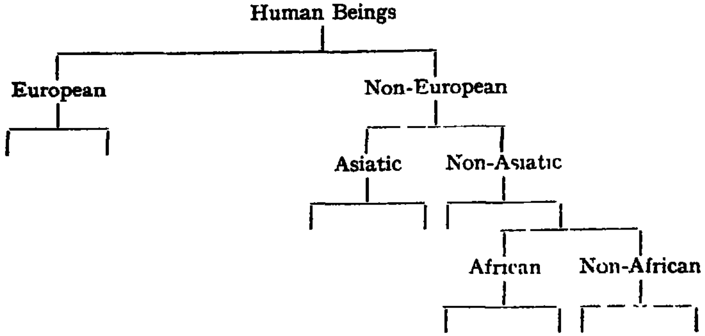
[1] The term 'universe' is adapted from the expression 'The universe of discourse'. Its sense here, therefore, differs fundamentally from that with which it was employed in Ch. IX., p. 170. The 'universe' in this usage is commonly a more restricted class than the universal class of the generalized class calculus.



Here, the principle of division is geographical, and the number of sub-classes at each step varies with the particular nature of the attribute considered. It is obvious that, if so desired, a manifold can always be reduced to a dichotomous division. Thus



The advantage of dichotomous division is that it secures, in an automatic way, conformity with the formal rules of division. It also facilitates certain elementary logical and statistical inferences *to which dichotomous division leads*.

The formal rules of division may be stated as follows :

1. Each step must be based upon a single principle of division.
2. Co-ordinate classes must be mutually exclusive.
3. Co-ordinate classes must be collectively exhaustive, i.e. co-extensive with the *summum genus*.

The breach of each of these rules may be illustrated by the Kantian classification of propositions quoted on page 46. The classification according to 'Relation' involves the double principle of division (*a*) according as the proposition is simple or compound, and (*b*) according as the proposition in question is categorically asserted or hypothetically entertained. A breach of the first rule involves in consequence a breach of the second.

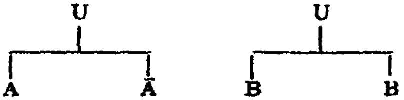
In the hypothetical proposition, *If p then q*, the components p and q are hypothetically entertained, and the fact that p implies q is categorically asserted. The classification is also not collectively exhaustive of the class of propositions as a whole.

Whilst such rules define the elementary formal conditions to which a classification should conform, they are obviously not sufficient to ensure that the classification so obtained should be of practical or scientific use. The practical or scientific requirements, however, do not admit of formal statement. They can only be defined by reference to the purpose the classification is intended to fulfil. The purpose suggests the appropriate principles of division. A classification of plants designed to assist identification would proceed on different lines from one designed to exhibit their important botanical features.

Among general scientific purposes prominence must be given to the desire to ascertain laws. Hence, as Mill remarks,[1] 'The ends of scientific classification are best answered when the objects are formed into groups respecting which a greater number of general propositions can be made, and those propositions more important, than could be made respecting any other groups into which the same things could be distributed'. And among the various possible principles of division according to which such classifications can be made, priority will, in general, be given to causal and genetic properties.

In this statement Mill has chiefly in mind absolutely universal propositions, propositions of the form All (or No) A's are B's. Scientific inquiry, however, is not restricted to such cases. The important question may be stated in a somewhat more comprehensive way.

Separate principles of division yield logically independent classes. For example, having classified the members of a population according as they are imbecile or not, and according as they are deaf-mutes or not, the class of imbeciles and the class of deaf-mutes are logically independent. The dual classification can be exhibited in the following form :



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[1] 'Logic', Bk. IV., Ch. VII., 2.

By combining two or more single dichotomous divisions of this kind we obtain more complex tables of the type shown on page 195. Conversely, any such complex tables may be resolved into simpler tables of this kind. The question arises: How are the logically independent classes related as a matter of fact? Though logically independent they may not be independent in fact. Membership of A may be connected with membership of B in such a way that the number of members in one class will provide a clue as to the number of members in the other.

An inquiry into the relations of factual dependence between two classes assumes one of three forms according to the nature of the information available with regard to each of these classes:

(i) It may be known concerning any individual in the universe with which we are concerned merely that it falls within the class A or in the class $\bar{A}$, and similarly that it falls within the class B or in the class $\bar{B}$. That is to say, we know that the property *a* and the property *b* are either present or absent. We do not know anything about the degree to which they are present. Thus, in the example already mentioned, we may know whether any individual in the given population possesses or lacks deaf-mutism, and whether he possesses or lacks imbecility. We do not know to what degree there is an impairment of normal powers of speech and hearing or the degree to which the intellectual faculties are defective. Such information may be lacking either because the attributes in question do not admit of measurement or because our sources of information are restricted in this respect.

(ii) It may be known that the properties *a* and *b* occur in certain specific forms a_1, a_2, . . . a_n, and b_1, b_2, . . . b_n, and that every individual in the universe possesses one of the specific attributes under *a* and one under *b*. In other words, the members of our universe admit of manifold classification on two independent principles. In consequence we have as many special classes combining the attributes *a* and *b* as can be formed from the combination of each of the classes A_1, A_2, . . . A_n with each of the classes B_1, B_2, . . . B_n. In this case our knowledge of the degree of dependence between the two major classes A and B may be rendered more precise. For example, we may be concerned with the possible interdependence between occupation and visual defect. We can classify a group of individuals, firstly, according as they are manual workers, clerks, managers, authors, etc.; and we can classify them again according as they manifest one or other of such defects as Myopia, Hypermetropia, and so on. We can then inquire how these visual defects are distributed among the occupations.

(iii) It may be found that the two properties a and b admit of precise measurement, i.e. that we can assert of each individual in our universe that he possesses the attribute a to such and such a degree, and that he possesses the attribute b to such and such a degree. The question of dependence is thus a question of the degree to which a measure of a affects the measure of b, and *vice versa*. Thus we may be concerned with the interdependence, in school work, of ability in scientific subjects and ability in literary subjects. We obtain as measures of these abilities the marks awarded in class examinations. Every pupil could be assigned a mark in science representing the average mark attained in the various science subjects. His literary ability might be measured in a similar way. The interdependence of the abilities is then revealed by the interdependence of these marks. It will be observed that this is in a sense a development of the principle of manifold classification. The sub-classes are defined by the properties corresponding to the units of the scale adopted. But when, as in this case, these sub-classes differ in a quantitative way, there are methods of determining interdependence which are not available in the simpler case of manifold classification previously considered.

In the first of these three cases the interdependence of A and B is described as '*association*' between A and B; in the second case it is described as a relation of *contingency*; and in the third case A and B are said to be *correlated*.

§ 2. Association and Contingency.

Association.—Factual, as contrasted with logical, dependence varies in degree. Two properties may be more or less independent, there may be a greater or a less tendency for one property to be accompanied by another. Translated into statements concerning the related classes, association between two attributes, a and b, will be reflected in the distribution of members of the class U between the four classes AB, $\bar{A}$B, A$\bar{B}$, $\bar{A}\bar{B}$. But t is important to notice that no statement concerning the association between *a* and *b* can be based simply upon the proportion of A's that are B, however great that proportion may be. If we know that 99 out of every 100 clerical workers suffer from some visual defect this does not prove that there is an association between clerical work and visual defect. It might be that 99 out of every 100 individuals in every other walk of life suffer from visual defect. The question of dependence is a matter of comparing the proportion of A's which are B with the proportion of non-A's which are B. It is only if there is some disparity between these proportions that association is indicated. Conversely, if two classes A and B are independent we may expect approximately the same proportion of A's among the B's as among the non-B's. If happiness is independent of the day of the week upon which marriage takes place we should expect to find approximately the same proportion of happy marriages occurring on a Friday as among those which occur on other days of the week. To prove that Friday is an unlucky day to be married on it is not sufficient to show that nine out of ten marriages occurring on this day are unhappy, it is necessary to show that a smaller proportion of unhappy marriages occur on other days. Hence we may state the general *criterion of independence: Two classes A and B are independent when the proportion of A's among the B's is the same as that among the non-B's.*

The number of members assigned to any class is called the class-frequency or the frequency of the class. Class-frequencies may be symbolized by placing the class symbols in brackets. AB is, in this way, a symbol for the class of things which have both the attributes *a* and *b*; (AB) is correspondingly a symbol for the *number* of things that are both *a* and *b*. In this symbolism the criterion of independence may be written

$$\frac{(\mathrm{AB})}{(\mathrm{B})} = \frac{(\mathrm{A\bar{B}})}{(\mathrm{\bar{B}})} \quad . \quad . \quad . \quad . \quad 1(a)$$

Stated in terms of the B's which are A

$$\frac{(\mathrm{AB})}{(\mathrm{A})} = \frac{(\mathrm{\bar{A}B})}{(\mathrm{\bar{A}})} \quad . \quad . \quad . \quad . \quad 1(b)$$

In the terms of the non-A's the formula would be

$$\frac{(\mathrm{\bar{A}B})}{(\mathrm{B})} = \frac{(\mathrm{\bar{A}\bar{B}})}{(\mathrm{\bar{B}})} \quad . \quad . \quad . \quad . \quad 1(c)$$

In terms of non-B's it becomes

$$\frac{(A\bar{B})}{(A)} = \frac{(\bar{A}\bar{B})}{(\bar{A})} \quad . \quad . \quad . \quad . \quad 1(d)$$

These four forms of the criterion of independence are equivalent. The relations between the class frequencies may be conveniently represented in a table of the following kind:[1]

	Attribute *b*.	Attribute $\bar{b}$.	Total.
Attribute *a*	(AB)	$(A\bar{B})$	(A)
Attribute $\bar{a}$	$(\bar{A}B)$	$(\bar{A}\bar{B})$	$(\bar{A})$
Total	(B)	$(\bar{B})$	(N)

Equations 1(*a*) and 1(*c*) state certain equalities which hold for the columns: 1(*b*) and 1(*d*) hold in the same way for the rows. The significance of this table may be illustrated by an inquiry into the relations between the class of individuals who are inoculated against cholera and that of individuals who escape cholera. Let A represent those who are inoculated and B those who are exempt from attack. The examined cases may be tabulated thus:[2]

	B (Exempt).	B (Attacked).	Total.
A (Inoculated)	276	3	279
Ā (Not-inoculated)	473	66	539
Total . .	749	69	818

Applying formula 1(*a*) we should find that if the class of those who were inoculated and the class of those who were exempt from cholera were independent, $\frac{AB}{B}$, i.e. $\frac{276}{749}$, would be approximately equal to $\frac{A\bar{B}}{\bar{B}}$, i.e. $\frac{3}{69}$. This is clearly not the case.

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[1] This table should be compared with those in Ch. VI., § 4.

[2] The data are from M. Greenwood and G. U. Yule: 'Proceedings of the Royal Society of Medicine,' Vol. VIII; and cited by the latter author in 'An Introduction to the Theory of Statistics,' Ch. III.

The class AB is 36·8 per cent. of the class B, whereas $A\bar{B}$ is only 4·3 per cent. of the class $\bar{B}$. A and B are, therefore, not independent.

The relation which obtains is, however, more clearly exhibited by formula 1(*b*). By this we can compare the percentage of the inoculated who were not attacked with the percentage of the not-inoculated who were not attacked. These are respectively 98·9 per cent. and 87·8 per cent. This serves incidentally to illustrate the importance of comparing the positive and the negative cases. The high percentage in the first case does not in itself show the benefit of inoculation. We could also compare the complementary proportions by formula 1(*d*). This shows that 1·1 per cent. of those who were inoculated were attacked but that 12·2 per cent. of those who were not inoculated were attacked By every comparison some measure of dependence is shown.

It is obvious that if the proportion of the A's among the B's is the same as among the non-B's the proportion of the A's among the non-B's is the same as in the universe as a whole. Consequently an alternative form of the criterion of dependence would be

$$\frac{(AB)}{(B)} = \frac{(A)}{N} \quad . \quad . \quad . \quad . \quad 2(a)$$

where N is the total number of individuals examined.

Similarly, the proportion of the B's among the A's will be the same as in the universe as a whole, i.e.

$$\frac{(AB)}{(A)} = \frac{(B)}{N} \quad . \quad . \quad . \quad . \quad 2(b)$$

The criterion of independence commonly assumes two other forms:

$$AB = \frac{(A)(B)}{N} \quad . \quad . \quad . \quad . \quad 2(c)$$

and

$$\frac{AB}{N} = \frac{(A)}{N} \cdot \frac{(B)}{N} \quad . \quad . \quad . \quad 2(d)$$

The last can be expressed in the form: *If the attributes a and b are independent the proportion of AB's in the universe is equal to the proportion of A's multiplied by the proportion of B's.*

Formula 2(*c*) yields corresponding equations for each of the classes, AB, $\bar{A}\bar{B}$, $A\bar{B}$, $\bar{A}B$, viz.,

$$(AB) = \frac{(A)(B)}{N}, \quad (\bar{A}\bar{B}) = \frac{(\bar{A})(\bar{B})}{N}, \quad A\bar{B} = \frac{(A)(\bar{B})}{N}, \quad \bar{A}B = \frac{(\bar{A})(B)}{N}.$$

These are of special importance, since they define the required frequencies for each of the four classes when A and B are independent. Consequently, it is useful to have a special symbol for the values which (AB), $(\bar{A}\bar{B})$, etc., must assume under conditions of independence. For this purpose the symbol $(AB)_0$ is employed. Thus $(AB)_0$ is the independence frequency for the class AB when A and B are independent. $(\bar{A}\bar{B})_0$ is the independence frequency for $\bar{A}\bar{B}$, $(A\bar{B})_0$ the independence frequency for $A\bar{B}$, and $(\bar{A}B)_0$ the independence frequency for $\bar{A}B$.

If A and B are not independent, then (AB) will be greater than $\frac{(A).(B)}{N}$ or less than $\frac{(A).(B)}{N}$. If $AB > \frac{(A).(B)}{N}$, A and B are said to be '*positively associated*' or simply '*associated*'. If $AB < \frac{(A).(B)}{N}$, A and B are said to be '*negatively associated*' or '*dissociated*'.

To say that A and B are positively associated means that the number of the A's which are B exceed the number to be expected if A and B are independent. To say that A and B are negatively associated means that the number of A's which are B falls short of the number to be expected if they are independent. As previously noticed, this is not a matter of the absolute number of A's that are B nor of the proportion of B's that are A, but is essentially a matter of the degree to which this proportion diverges from the equation of independence.

It is, of course, not to be expected when the two classes are independent that the proportion of the A's among the B's will be *exactly* equal to the proportion of the A's among the not-B's. There will be slight divergences due to 'chance' or more strictly to the 'fluctuations of sampling'. Statistical theory enables a distinction to be drawn between these small degrees of association and those which are 'significant'.

The fact that the association or dissociation between two classes admits of degrees suggests that the degrees of association may admit of measurement. Various measures have been suggested. It is easy to show that when A and B are independent

$$(AB) . (\overline{AB}) = (A\bar{B}) . (\bar{A}B).$$

That is, the number of cases in which both attributes are present multiplied by the number in which they are both absent is equal to the number in which the first is present and the second absent multiplied by the number in which the first is absent and the second present.

If, further, the two attributes are completely associated either $(A\bar{B})$ or $(\bar{A}B)$ will be zero: if they are completely dissociated (AB) or $(\overline{AB})$ will be zero. From this consideration the following formula has been derived:

$$Q = \frac{(AB)(\overline{AB}) - (A\bar{B})(\bar{A}B)}{(AB)(\overline{AB}) + (A\bar{B})(\bar{A}B)}.$$

'Q' is called the coefficient of association. It becomes 0 by the above equation when the attributes are independent. It becomes 1 when the two classes are completely associated, and − 1 when they are completely dissociated. Every degree of association and dissociation is thus represented on a scale which ranges between + 1 and − 1. In this scale the zero point represents independence. The formula may, perhaps, be more readily understood in the following form:

$$Q = \frac{ad - bc}{ad + bc},$$

where a = the number of individuals belonging to both classes.

b = the number of individuals which belong to the first but not to the second class.

c = the number of individuals which belong to the second but not to the first class.

d = the number of individuals which belong to neither class.

Contingency.—A coefficient of association is a measure of the degree of interdependence between two classes yielded by separate principles in a dichotomous division. The 'coefficient of contingency' provides an analogous measure for manifold divisions.

Suppose the class U is divided first into species of A's, A_1, A_2, . . . A_n; and again into species of B's, B_1, B_2, . . . B_n.

The population of a cosmopolitan city might, for example, be classified according to nationality and again according to occupation. We might then proceed to inquire whether nationality and occupation are independent or to any degree connected.

In dichotomous division we obtain 2 × 2 sub-classes. In a manifold division in which a has three species and b four species we obtain 3 × 4, i.e. 12 sub-classes. If a has n species and b has m species we obtain $n \times m$ sub-classes. These can be arranged in a '*contingency table*':

	a_1	a_2	a_3	...	a_n	Total
b_1	(A_1B_1)	(A_2B_1)	(A_3B_1)	...	(A_nB_1)	(B_1)
b_2	(A_1B_2)	(A_2B_2)	(A_3B_2)	...	(A_nB_2)	(B_2)
b_3	(A_1B_3)	(A_2B_3)	(A_3B_3)	...	(A_nB_3)	(B_3)
...	...	...	...	...	...	...
b_m	(A_1B_m)	(A_2B_m)	(A_3B_m)	...	(A_nB_m)	(B_m)
Total	(A_1)	(A_2)	(A_3)	...	(A_n)	N

As before the frequency of a class in any 'cell' is shown by enclosing the class symbol in brackets. (A_1B_1) represents the number of individuals in the class A_1B_1; (A_2B_3) similarly represents the frequency of the class A_2B_3; and (A_nB_m) the frequency of the class A_nB_m.

It was seen, in the case of association, that if the two classes A and B are completely independent there is a certain frequency to be expected in each of the cells, AB, $\bar{A}\bar{B}$, $A\bar{B}$, $\bar{A}B$. Similarly, in the contingency table, if A and B are independent there is an 'independence frequency' for each cell. This is given by the formula: For all values of m and n

$$(A_nB_m) = \frac{(A_n)(B_m)}{N}.$$

This independence frequency may be symbolized

$$(A_nB_m)_0.$$

If, on the other hand, A and B are in some way connected there will be a difference between the *observed* frequency and the *independence* frequency. In the example cited if nationality and occupation are not independent we shall find either more or less Greeks who are clerical workers than was to be expected, or more or less German musicians than was to be expected, etc.

These differences may be written

$$\delta_{nm} = (A_nB_m) - (A_nB_m)_0.$$

The value of δ_{nm} for any cell, may, of course, be positive or negative. It is described as the 'contingency' of the cell.

It is clear that any measure of the degree of interdependence between A and B must take into account the values of δ for all the cells in the table. Since the magnitude of δ is equally important irrespective of the sign, differences of sign must be eliminated. This is best effected by squaring the values of δ. Further, the measure must be based not on the absolute value of δ^2 but upon the ratio of this value to the corresponding value of $(AB)_0$. By summing these ratios we obtain a measure symbolized by χ^2. Employing the symbol Σ for 'the sum of all quantities like' we obtain the formula

$$\chi^2 = \sum\left(\frac{\delta^2_{nm}}{(A_nB_m)_0}\right).$$

If A and B are independent χ^2 will be zero because every value of δ^2 will be zero. A further formula has been constructed in accordance with which all departures from zero will be expressed by a number somewhere between 0 and 1. This is called the *mean square contingency coefficient.* It is expressed thus:

$$C = \sqrt{\frac{\chi^2}{N + \chi^2}}.$$

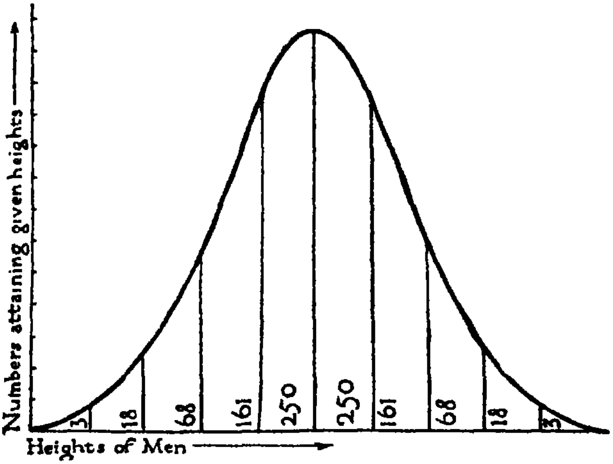
C is thus a measure of the degree to which A and B are connected. If they are independent $C = 0$. The more closely they are connected the more closely will C approximate to 1. Strictly, however, this is only the case when the sub-classes are fairly numerous. It is only when the table is 5 × 5-fold or over that the maximum value of C approximates to ·9. The formula is, therefore, best employed only in cases in which the division is 5 × 5-fold or greater. Under such conditions the value of C is comparable with other measures of dependence. This measure, unlike the coefficient of association, does not vary in sign. But in certain cases a sign can be arbitrarily employed to indicate a difference in contingency analogous to the difference between association and dissociation.

§ 8. Distribution and Correlation of Attributes.

So far, we have been concerned with relations of dependence and independence between two classes when all that is known with regard to any individual in the universe in question is either (*a*) that it possesses or lacks the attributes by reference to which the two classes are defined, or (*b*) that it possesses one or other of the species of each of these attributes. We have now to consider the case in which we also know the *degree* to which the relevant attributes are present.

It is possible to form a class in which every member possesses a given attribute to precisely the same degree, but we are commonly concerned with classes in which the defining attribute is present in varying degrees. In this case it is desirable to know how the attribute in question is distributed.

The most common form of distribution is one that may be graphically represented in a diagram of the following kind:



In this mode of representation the varying degrees to which an attribute may be present (i.e. the various values which may be assigned to a measure of the attribute) are represented by a scale on the base line. The distance from the base line of any point on the curve represents the number of individuals possessing the attribute to the degrees indicated on the scale below. The figure here given shows the theoretical distribution of heights for 1000 men. Again, let the diagram represent the distribution of height of 1000 12-year-old boys. It might be found, say, that 150 of these were 142 cm. tall, that 62 were 150 cm., that 23 were 154 cm., that 5 were 158 cm., and only one boy was over 158 cm. The decreasing number of cases for each increase in height is shown by the form of the curve. Similar decreases would be found for variations in the opposite direction. In such a case there is some central measure (here 142 cm.) around which the frequencies of other measures are symmetrically distributed. This kind of symmetrical distribution is described as ' normal ', and the curve representing it ' the normal curve of distribution '. It represents in a simple manner most of the important properties of the class to which reference is made.

In describing a class, however, it would be confusing always to enumerate the frequencies of each value of the variable attribute, and graphical representation is not always convenient. It is, therefore, desirable that the essential properties of the class should be summed up in a small number of determinate pieces of information. It will be found that three items are in general sufficient. We require to know :

i. The form of the distribution.
ii. Some central measure or measures.
iii. How other values are distributed in relation to the central measures.

The form of distribution may be reduced to one of a small number of types. The most common is the ' normal ' form above described. One departure from this is that of the ' skewed ' distribution, i.e. when the apex of the curve is either to the right or the left of the central value in the scale. For example, if we classified a population by income we should find that there was an enormous preponderance of cases of small incomes. The apex would not be half-way between the point representing the richest man in the community and the poorest. It would be much nearer the latter. If we classified the individuals according to age at which they died we should find a skew in the direction of the higher measures of age.

Another possible, but not common form of distribution, is that in which the mid-point represents the smallest frequency in the class. We thus obtain a U-shaped curve. It is clear that in skewed or otherwise abnormal distributions we require rather more information with regard to its characteristic measures. We may, therefore, in what follows, restrict attention to the case of normal distribution.

Measures of central tendency, in the case of normal distribution, must obviously serve the purpose of exhibiting the most representative members of the class. For this purpose we may employ the *Mean*, the *Median* or the *Mode*.

The mean, i.e. *the ordinary arithmetical mean* or *average* (M) is a familiar measure. It is computed by dividing the sum of the measures of the variable attribute (m) by the number of cases (n). Hence it may be defined:

$$M = \frac{\Sigma m}{n}.$$

In this way we should compute the average weekly income of a group of individuals by adding together the weekly incomes of each and dividing the sum by their number. It will be seen later that this measure, though generally useful, requires to be supplemented by further information.

The *median* is the midmost member of a series of values arranged in order of magnitude. This is a convenient measure when we are interested in the position of an individual in a rank. The formula is:

$$\text{Median} = \text{the } \frac{n+1}{2}^{\text{th}} \text{ value.}$$

In a list of 99 values the median would be the 50th. It would have 49 values above and 49 values below. When the number of cases is even the value of the median is somewhere between two actual values. In 100 cases it is between the 50th and the 51st.[1]

The *mode* is the measure which occurs more frequently than any other. In some cases there may be more than one mode. In normal distributions it is very close to the mean and the median, but in some cases, e.g. in skewed distributions, it may be very far removed from the average and the median. The mode alone of the three measures of central tendency represents a measure actually obtained rather than computed. For this reason it is very useful for certain purposes. If the average family in a certain community consists of 3·2 children it is clear that this cannot represent any actual case. The number of children in any family must be represented by a whole number. If 25 per cent. of the families contain 2 children, 40 per cent. 3 children, 38 per cent. 4 children, and so on, the 'modal family' has 3 children. In the same way the fact that the average boy of 12 is 142·9 cm. tall does not imply that any actual boy is of this stature. The most frequent measure, i.e. the mode, may be, say, 142·5 cm., where measures are accurate to ·5 cm.

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[1] If the 50th value is 120 and the 51st is 121 the median is not necessarily 120·5. The method of computing the true median may be found in any elementary textbook of statistics.

For most purposes it is not sufficient to know the general tendency in a group of measures. An average, median or mode in isolation may be seriously misleading. This is obviously the case when distribution is abnormal. The average income of a population might be £5 per week. This in relation to a given index of the cost of living might convey the impression of universal comfort. The average, however, might be derived from figures which revealed a small portion of the population as possessed of extreme wealth and the remainder on the verge of starvation. Even where distribution is normal an average is of little significance if not accompanied by some indication of the range of the values from which it is computed. An average income of £5 per week might represent a central point in a series of measures ranging between £4 and £6 or in a series ranging between zero point and £10. A measure of central tendency should, therefore, always be accompanied by a '*measure of dispersion*'. Several such measures are available.

One of the simplest ways of indicating the range of variation is to specify the extreme cases. This is better than nothing, but it is for many reasons unsatisfactory. The most useful measures of dispersion are the *Average Deviation*, the *Standard Deviation* and the *Quartile Deviation*.

The *average deviation* (or the *mean deviation* as it is sometimes called), is the arithmetical mean of the separate deviations, d, from some measure of central tendency taken without regard to sign. Unless otherwise specified these deviations are measured from the mean. It may thus be symbolized:

$$\text{A.D.} = \frac{\Sigma d}{n}.$$

If, for example, 10 boys of 12 years of age measure respectively 136, 137, 138, 139, 140, 141, 142, 143, 144 and 145 cms., their mean height is 140·5 cm. Their respective deviations from the mean are 4·5, 3·5, 2·5, 1·5, 0·5, 0·5, 1·5, 2·5, 3·5 and 4·5. The sum of these deviations is 25·0 cm.; and the average deviation is thus 2·5 cm. The facts with regard to these ten boys are summed up in the statement that they have an average height of 140·5 cm. with an average deviation of 2·5 cm. This measure of dispersion is easy to compute, but is not the most useful when the data require algebraical treatment. For most statistical purposes the Standard Deviation is employed.

The *standard deviation*, represented by S.D. or σ, is the square root of the average of the squares of the individual deviations:

$$\sigma = \sqrt{\frac{\Sigma(d^2)}{n}}.$$

It yields a mean of dispersion somewhat larger than the average deviation; but when computed from large numbers and for normal distributions the two measures stand in a constant relation, viz.

$$\sigma = 1{\cdot}2533 \text{ A.D.}$$
$$\text{A.D.} = 0{\cdot}7979\ \sigma.$$

The *quartile deviation* or *semi-interquartile range*, (Q). provides a third convenient measure of deviation. If the measures of the variable are arranged in order of magnitude we may divide them, as we have seen, into two equal parts by the median. By equally dividing each half again we obtain (1) a measure Q_1 such that one quarter of the values lie below this value and three-quarters above it, and (2) a measure Q_3 such that one-quarter of the values lie above this value and three-quarters below it. The range is thus divided into four '*quartiles*' by the median, Q_1, and Q_3. That is to say, the total class is divided into four equal sub-classes. If M_i be the median in a symmetrical distribution

$$M_i - Q_1 = Q_3 - M_i,$$

and this difference provides a convenient measure of dispersion—the semi-interquartile range.

Since in practice distributions are never perfectly symmetrical we may take the mean of the two deviations. Thus

$$Q = \frac{Q_3 - Q_1}{2}.$$

Where the deviations are due to the chances of fluctuations and the distribution is symmetrical this measure is described as the *Probable Error* (P.E.). It defines, that is to say, a quantity such that we may expect greater or less deviations from the median with about equal frequency. Any case selected by chance is as likely to fall within the interquartile range as it is to fall outside. It is related to other measures of dispersion in a manner shown in the formula:

$$Q \text{ (or P.E.)} = 0{\cdot}6745\sqrt{\frac{\Sigma(d^2)}{n}},$$

i.e.

$$Q = 0{\cdot}6745\ \sigma.$$

Roughly, it may be said, the semi-interquartile range is about two-thirds of the standard deviation.

We are thus able to summarize the most important facts with regard to a class of the type with which we are concerned when we are able (1) to specify as normal its form of distribution, (2) to specify some measure of central tendency, such as the mean, the median or the mode, and (3) to specify some measure of dispersion, such as the average deviation, the standard deviation or the semi-interquartile range.

The general properties of a class in which an attribute is distributed normally are of immense importance when two such classes are compared with a view to ascertaining relations of dependence. Such a comparison may be made by a refinement of the table of contingency.

Let us suppose that we are concerned with the relation between age and health. We could construct a contingency table by classifying the members of a population—or, for simplicity, a large institution such as a factory—firstly according to age, and secondly according to health. We could divide them into groups such as Youths, Young Men, Middle-aged Men, etc., and into groups such as Very Healthy, Liable to Slight Indisposition, those who have suffered from Serious Illnesses, etc. It is, however, obviously more satisfactory if we can reduce these differences to quantitative terms, and construct the sub-classes in such a way that between each class and the next there is the same degree of difference. As a first approximation to exactitude we could divide the individuals into age groups, 15, 16, 17, 18 years, etc., and as a rough measure of health divide them again according to the number of days per year that they were absent from work on account of sickness.

The contingency table then would show a series of equidistant age groups, $A_1, A_2, \ldots A_n$, and a corresponding series of equidistant groups, $B_1, B_2, \ldots B_n$, defined in terms of days absent from work. In any such table the three important characteristics of the two classes concerned would be reflected by the manner in which the individuals were distributed among the various cells. In addition, this distribution will also indicate to what extent the two classes may be interdependent. We may consider three cases:

(i) The two attributes are normally distributed and are independent.

In this case it will be found that the distributions in all parallel 'arrays' are similar to one another, and are similar to the distribution in the universe as a whole. That is to say, every row will reflect the same proportions in corresponding cells. Each row will be similar to the totals of the A's shown at the bottom of the table. Similarly, each column will show a similar distribution, and each will be similar to the totals of the column of B's shown on the right. Each row will reflect the normality of the distribution by exhibiting increasing numbers in successive cells up to a certain point and then a symmetrical decrease on the other side. Similarly with the columns. It follows that the means of the rows will all appear in the same column, and the means of the columns in the same row. In other words, the line of means of the rows will be at right angles to the line of means of the columns. And, on each side of these two lines of means, there will be similar lines for quartiles and other measures of dispersion. Such a distribution is characteristic of independence.

(ii) The two attributes are normally distributed but are directly interdependent.

When one attribute depends upon the other it will be found that the means of parallel arrays no longer remain at right angles. It is easy to see why this should be the case. If a low value of the attribute *a* generally accompanies a low value of the attribute *b*, and if the higher the value of *a* the higher the value of *b* we shall find the cells on the line A_1 are more crowded on the left than on the right. In a corresponding way we shall find relatively few cases in which a low value of *b* is associated with a high value of *a*. Among the high values of *a* we shall find an increasing number in each cell as we pass to the right. Consequently, for the low values of *a* the mean will be correspondingly deflected to the left and for high values of *a* it will be deflected to the right. It will occupy intermediate positions for intermediate values of *a*.

In the same way the low values of *b* will be crowded upwards on the left-hand of the table and downwards on the right-hand side. In consequence we find that with this form of dependence the two lines of means instead of being at right angles converge towards the diagonal which runs from A_1B_1 to A_nB_m. In the limiting case they unite in this diagonal. This, however, can only arise if the dependence is so close that for each value of *a* there is only one value of *b*, i.e. when the measure of dispersion is reduced to zero. This relationship constitutes a perfect *positive correlation*.

(iii) The two attributes are normally distributed but are inversely related.

Here a low value of *a* is combined with a high value of *b*, and vice versa. It is easy to see, by parity of reason, that in this case the two lines of means converge but in the opposite direction. They converge to the diagonal which runs from A_nB_1 to A_1B_m. Consequently, this distribution would reflect a perfect inverse correlation.

The systematic presentation of the data in this way, when the numbers in each cell are specified, enables us to see at a glance whether the two properties are interdependent, and to gauge to some extent the degree to which they are interdependent.

But as in the case of association and the simpler form of contingency it is desirable to have a measure of this relationship. For this purpose a *coefficient of correlation* (*r*) has been devised, the value of which in any case can be determined from the formula[1]

$$r = \frac{\Sigma(xy)}{N\sigma_1\sigma_2}.$$

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[1] Considerable space, though no very advanced mathematics, would be required to exhibit the way in which the formula is derived. For information on this point the student may refer to one of the works mentioned in the bibliography at the end of the chapter.

The x's and y's from which the numerator is obtained are the series of measures of the deviations from the mean for each of the individuals of the classes compared. The x's are the deviations of each individual A from the mean of the A's and the y's are the corresponding deviations from the mean of the B's. N, as before, is the total number of cases; σ_1 and σ_2 are the standard deviations of the two groups.

The formula is so constructed that it yields for r the value 0 when the two lines of means in the contingency table are at right angles, the value + 1 when they converge in the manner required for a perfect positive correlation and − 1 for perfect negative correlation. The various degrees of correlation are thus represented by a numerical measure ranging between + 1 and − 1, with zero as the point of independence.

It is important to observe that coefficients of association, of contingency and of correlation are never proof of *direct* causal relations between the properties, the interdependence of which they serve to exhibit. It may be that poverty is 'associated' with intellectual defect. This does not prove either that poverty (involving ill nutrition and other disabilities) retards intellectual development, or that intellectual defect prevents the accumulation of wealth. Both hypotheses are plausible, but the coefficient does not help us to decide between them. It may be the case that both poverty and intellectual defect are joint effects of a common cause, and even in this way connected only in a very indirect manner.[1] Similar possibilities are open in the case of relations established by the coefficient of contingency and the coefficient of correlation.

It is also important to notice that these coefficients are subject to a margin of uncertainty, for the measure of which certain additional formulæ are required. Zero correlation is in fact extremely rare. Purely chance distributions can produce an appearance of correlation, especially when the number of examined instances is small. If $N = 100$ a coefficient of $r = 0{\cdot}3$ might occasionally be produced by the chances of sampling. In general the smaller the number of cases the higher the coefficient must be in order to be 'significant'. Further statistical methods must be employed to enable us to decide what reliance may be placed upon the measures obtained.

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[1] Throughout this chapter the term 'interdependence' has been employed merely to signify *some* or other relation of dependence direct or indirect between the terms.

In the case of every statistical measure of this kind it is also to be remembered that they are at best abstracts from the data. They can never give more information, and they generally give less, than is contained in the tables from which they are derived. An intelligent survey of the tables themselves will generally yield an impression which, though in a sense not so 'precise', is considerably more reliable. It is from the neglect of such considerations that the impression has got abroad that 'statistics can prove anything'.

§ 4. Probability.

In the foregoing sections we were concerned with some of the developments from elementary logic which become possible when we supplement the traditional propositional forms by others which admit of the explicit 'quantification of terms'. Such possibilities have from time to time been noticed by logicians, but systematic study of these developments has been left in the main to the statisticians.

A further important extension, and one that is properly logical, relates to judgments of probability. We frequently meet statements of the form: *S is probably P*, *It is probable that All S is P*, *If P is true then Q is probable.* It is evident that such assertions may be true or false, rational or prejudiced, well or ill-grounded. Though in a sense expressive of 'subjective states' and degrees of uncertainty they also depend upon constitutive conditions. They are subject to logical criteria, and to formulate these criteria is a matter, not only of practical importance, but of considerable theoretical interest.

The study of probability has a philosophical and a formal aspect. The question of the analysis of the concept of 'probability' is a philosophical problem. So, too, perhaps is the formulation of the ultimate conditions requisite for non-derivative judgments of probability. But given certain primitive concepts and primitive propositions a technical calculus of probabilities may be developed in a purely formal way. A third department of the general theory of probability relates to its practical applications. The general theory itself lies behind the theory of statistics, but statistical theory provides one of the principal fields in which this general theory is required. Propositions concerning the distribution of attributes in a class, or concerning the frequencies with which certain attributes are combined, constitute one of the chief sources of probable judgments. If we know that three out of every four A's are B's, we may proceed to assert that if any one of these A's is selected at random the probability that it will be B is $\frac{3}{4}$. Given that the mean height of the men in a group is 5 ft. 8 ins., and that 5 ft. 6 ins. to 5 ft. 10 ins. constitutes the interquartile range, the chance that any member of the group selected at random is over 5 ft. 10 ins. tall is one in four.

Analysis.—The question of the analysis of the concept of probability is still a matter in dispute, but certain points are fairly clear.

Commonly, we speak of the probability, or the chance, of a certain event happening. 'It will probably rain', we say; thereby suggesting that probability is a character of events. It is fairly obvious, however, that this kind of assertion is derivative from one concerning the probability that a certain *proposition* is true. '*It will probably rain*' means '*That it will rain is probable*'. Probability is primarily an attribute of propositions. Some writers tend to oppose *probability* and *truth*; but this is only legitimate if by 'truth' *certainty* is meant. In the common and natural sense of 'true', truth and probability are independent attributes. A proposition may be both probable and true, improbable but true, and probable but false.

Since probability is an attribute of propositions it follows that all propositions in which it occurs are secondary propositions or propositions of some higher order. *S is probably P* ≡ *S is P is probable*. *S is P is probably false* is a proposition of the third order, i.e. $\{(S \text{ is } P) \text{ is false}\}$ *is probable*.

Given that in probable judgments we are always asserting something about a proposition, what is it that we are asserting? According to the analysis of Mr. Keynes[1] it is something of this kind. In asserting that p is probable to degree, d, we are asserting that there is some hypothesis, h, so related to the conclusion p that knowledge of h justifies belief in p to the degree d. For example, we say that the probability of drawing a black ball from a certain bag is $\frac{1}{2}$. This means that there is some body of data, *h* (e.g. that the bag contains an equal number of black and white balls well sorted and indistinguishable to touch, etc.), such that, relatively to this body of evidence, it is reasonable to adopt to the proposition *A black ball will be drawn* the attitude of mind which lies midway between certainty that it is true and certainty that it is false. Similarly, in making the more indeterminate assertion that '*It may rain*', i.e. '*It is not improbable that it will rain*', we are asserting that there is *some* body of evidence so related to the proposition that we are rationally obliged to adopt a certain attitude which lies *somewhere* between the certainty that the proposition is false and the certainty that it is true.

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[1] 'Treatise on Probability,' Chs. I. and II.

Two points, in particular, call for notice in this account of the matter. First there is an inexpungible reference to an attitude of mind on the part of some thinker. Reference is made to what is described as 'degrees of belief'. It is asserted that a certain degree of belief is rational, or that it is 'appropriate'. This is not identical with the purely psychological assertion that any thinker would adopt this degree of belief. It is the 'normative' assertion that such a degree of belief is the '*correct*' degree under the circumstances.

Secondly, there is an inexpungible reference to a relation between the probable proposition and some evidence or hypothesis. In other words, 'probable' like such predicates as 'tall' or 'distant' is relational. As we say simply that '*X is distant*', meaning it is distant *from somewhere*, so when we say that '*p is probable*' we mean that *p* is related in a certain way to something else unspecified. It is not simply that reference to evidence is required to make the assertion of probability *valid*; it is required to make it *significant*. A proposition in isolation cannot be probable any more than a man in isolation can be a neighbour or a servant.

In consequence of this reference to the mental attitude of the thinker and to the evidence upon which the judgment is based Mr. Keynes speaks of the term 'probability' as being applied in three senses, (1) to the proposition itself, (2) to the degree of rational belief, and (3) to the relation between the conclusion and the evidence. More strictly, perhaps, we should say not that there are three senses of the term, but that the proposition, the reference to the degree of belief, and the reference to evidence are three *elements* in the analysis of a proposition to the effect that so and so is probable. A proposition is probable, and this means that it is to be accepted with a degree of belief appropriate to its relations to the evidence to which reference is explicitly or implicitly made.

Within the scope of this formula different views may be taken concerning the nature of 'degrees of belief', and with regard to the nature of the relation between the evidence and the conclusion upon which the appropriate degree of belief depends. It is not necessary to suppose that there is any direct unanalysable relation of *probability* between the evidence and the conclusion. So far as any such irreducible relation is involved at all it is plausible to suppose that it is of the 'multiple type'. This might be expressed in the form: *h suggests to degree d that p is true.*

The general form of any assertion of probability is thus $p/h = d$, i.e. a set of propositions h justifies a rational belief in a proposition (or set of propositions) p to the degree d. Under such conditions we may for brevity say that there is a probability relation of degree d between p and h. The expression 'p/h' is read 'p upon h'.

The expression *a probability relation between p and h* need not be interpreted as implying that there is any unique and specific diadic relation between p and h. The probability relation may be simply described as *any* relation which does justify rational belief to such and such a degree. All sorts of relations may serve this purpose. In general such relations will be analogous in some respects to the relation of entailment. If h entails p then the acceptance of h justifies the unqualified acceptance of p, i.e. $p/h = 1$, where 1 represents certainty. If h entails $\bar{p}$ then $p/h = 0$, where 0 represents 'contra-certitude', i.e. the certainty that p is false. In terms of this notation all degrees of probability intermediate between certitude and contra-certitude can theoretically be represented by a fraction. There is, however, an important difference between entailment and probability relations. If h entails p and we know h we can then categorically assert p without reference to h. In assertions of probability, however, reference to the hypothesis can never be ignored. The assertion '*p is probable*' is not significant without reference to some hypothesis. In common parlance we do in fact *state* that something is probable without explicit reference to the evidence; but we cannot *judge* that it is probable without some such reference. Such statements can be interpreted as meaning that p is probable in the light of certain evidence unspecified, i.e. that there is some evidence on which p is probable.

Though every judgment of probability expresses a degree of rational belief between the extremes signified by 0 and 1 it does not follow that every such judgment is to the effect that some proposition p has such and such a *determinate* degree of probability. Many probable judgments are *indeterminate.* Commonly the statement that '*p is probable*' merely states elliptically that *p is more probable than $\bar{p}$.* For general logical purposes it is more convenient to use the expression *p is probable,* not to mean that it is more likely true than false, but to mean simply that it has *some* degree of probability. This is the limit of indeterminateness. Probable judgment can be ranged between this and the limit in which some quite determinate probability is assigned. Determinate probabilities alone are expressed by a fraction. Even when a specific measure of probability cannot be assigned, we can nevertheless often compare probabilities as greater or less; and rationally judge that the probability of a certain proposition is increased by certain evidence. We know, for example, that it is more probable that the sun will shine to-morrow than it is that it will rain to-morrow, even though we cannot assign a determinate probability to either proposition. If the barometer is falling and the clouds are gathering we know that the proposition 'It will rain to-morrow' is more probable than it was; though we cannot say *how much* more probable it is.

For scientific and practical purposes some measure of probability is required. Three possibilities require to be considered. (1) We may assign a specific numerical measure. This is in general only possible when the probability arises from a determinate number of equiprobable alternatives. In this way we assign the chance of a die falling 'five' uppermost as one in six. (2) We may compare the probability of a proposition with the probability of its contradictory, and assert that it is either greater or less than this. This is the kind of measure upon which action is usually taken. We cannot assign a definite measure either to the probability of rain or of sunshine, but when we can assert that rain is more likely than not we can decide to take an umbrella. (3) A third possibility, of some theoretical and practical interest but one that has not been systematically studied, lies in the comparison of a given probability with a standard or scale. Such comparisons occur in daily life. We reason as follows: *Will X visit us to-day? Although it may quite possibly rain, and he will not come if it rains, nevertheless he came last week when the weather was even more uncertain than it is to-day.* In this argument the relative probabilities of rain and no rain are not compared, but the probability of rain to-day is compared with the probability of rain as it was estimated on some prior occasion. On the basis of this argument it would be reasonable to expect X.

Formal Logic of Probability.—It is obvious that elementary judgments of probability may arise in connection with propositions of the traditional form. All S is P; ∴ Probably All P is S.[1] If P then Q, and Q; ∴ Probably P. All M is probably P, All S is probably M; ∴ All S is probably P. Such inferences, however, are very trivial and become important only when evidence is provided in a form which enables more or less determinate probabilities to be assigned.

A purely formal treatment requires a more systematic approach and the construction of a calculus analogous to the calculus of propositions outlined in Chapter IX. Such a generalized theory would start with primitive ideas, definitions and primitive propositions, and from these deduce general theorems. We require definitions of the properties of probability relations, of equivalence and consistency, and of addition and multiplication. Fundamental among the axioms are those of addition and multiplication.

The axiom of addition asserts that the probabilities of an alternative proposition, *p or q*, is equal to the *sum* of the several probabilities. Formally stated:

$$\textit{Prob.}\ (p \textit{ or } q) = \textit{Prob.}\ p + \textit{Prob.}\ q.$$

The axiom of multiplication asserts that the probability of the conjunctive proposition p and q (when p and q are independent) is equal to the product of the probabilities of p and q individually, i.e.

$$\textit{Prob.}\ (p \textit{ and } q) = \textit{Prob.}\ p \times \textit{Prob.}\ q.$$

This is derivative from a more general axiom which is applicable to the case in which the probabilities of p and q are not independent, viz.

$$\textit{Prob.}\ (p \textit{ and } q) = \textit{Prob. of } p \times \textit{Prob. of } q \textit{ given } p.$$

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[1] Here and henceforth it must be remembered that p is probable means only that p has some probability, not that it is more likely to be true than false.

This form covers the case in which knowledge of one proposition influences the probability of the other. Given that there are two black balls and one white in a bag, what is the probability that in both of two successive draws a black ball will be selected? Let p be the proposition: *The first ball drawn is black*, and q the proposition: *The second ball drawn is black.* The probability of p is $\frac{2}{3}$. What is the probability of q? It all depends upon whether the first ball is returned before the second draw is made. If it is returned the probability of the second draw is the same as the first, if it is not returned (and the first ball drawn is black) then the probability of q is $\frac{1}{2}$.

The two axioms can, of course, be extended to cover many alternants and many conjuncts, and they provide the fundamental principles from which many complex theorems are derived. We may consider briefly, for illustration, some elementary computations.[1]

The axioms apply to the cases in which we commence with probabilities, and the problem is simply that of determining derivative probabilities. A simple case is that in which in throwing dice we wish to calculate the probability of any one falling with a given face upwards, or to calculate any combination which is antecedently possible.

We throw a single die of normal unbiassed construction under normal conditions. What is the probability (1) that the face bearing one spot will be uppermost? (2) that the number uppermost will be greater than *one?* (3) that it will be an odd number? (4) that it will not be an odd number?

Case (1).—We know (*a*) that the uppermost face must be either *one* or *two* or *three* or *four* or *five* or *six*, we know (*b*) that it will not be more than one of these, and (*c*) we have no reason for assigning a higher probability to one of the alternatives than to any other. Under these conditions the certainty that one or other face will be uppermost is distributed into six equi-probable alternatives. Of these we select one. The probability of this is $\frac{1}{6}$.

Case (2).—The conditions are similar and our antecedent knowledge is the same. The proposition that a number higher than one will be uppermost is equivalent to the alternative:

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[1] These cases belong, of course, to the arithmetic as opposed to the algebra or the logic of probability.

Either two or three or four or five or six will be uppermost. The probability of each is $\frac{1}{6}$. Their sum is thus $\frac{5}{6}$.

Case (3).—The proposition is here equivalent to the suggestion that either *one* or *three* or *five* will be uppermost. We have, therefore, to sum three equal probabilities each of which has the value $\frac{1}{6}$. The result is that the probability is $\frac{1}{2}$.

Case (4).—This is precisely similar and yields the same value.

It will be noticed that cases (1) and (2) are contradictories and cases (3) and (4) are contradictories. The probability that one or other of these cases will be realized is in both cases 1. The probability that neither will be realized is 0. This result is in accordance with the definition of certainty and impossibility. In the second pair of contradictories each is equi-probable. This is not the case with the first pair. This is a point of some importance. Both cases would be expressed in the form: *What is the probability of p? What is the probability of non-p?* To this question we see that the answer depends upon what p happens to be. We cannot assume, as has sometimes been assumed with disastrous consequences, that the alternatives p and *not-p* are equally probable. They are equally probable when they can be analysed into an equal number of equi-probable ultimate alternatives. This is by no means commonly the case.

Next consider the case in which we throw two dice. What is the probability (i) that both will fall *one* uppermost? (ii) that one or other but not both will fall in this way? (iii) that one will be odd and the other even?

Case (i).—The probability that the first dice will fall one uppermost is $\frac{1}{6}$. The probability that the second dice will fall similarly (being independent) is $\frac{1}{6}$. By the axiom of multiplication, the probability of both falling *one* uppermost is $\frac{1}{36}$.

Case (ii).—The probability that one or other but not both will fall one uppermost is the sum of two possibilities, viz. (*a*) that the first will fall one uppermost and the second will not, (*b*) that the first will not and the second will. The probability of the first is clearly $(\frac{1}{6} \times \frac{5}{6})$ the probability of the second possibility $(\frac{5}{6} \times \frac{1}{6})$. Hence the required probability is

$$(\tfrac{1}{6} \times \tfrac{5}{6}) + (\tfrac{5}{6} \times \tfrac{1}{6}), \text{ i.e. } \tfrac{10}{36}.$$

Case (iii).—The probability that one will turn up odd and the other even is similarly

$$(\tfrac{1}{2} \times \tfrac{1}{2}) + (\tfrac{1}{2} \times \tfrac{1}{2}) = \tfrac{1}{2}.$$

Applications.—The applications of the principles of probability are obviously many and important. They are important not only for the calculation of betting odds and insurance rates, but also, in their more general forms, for the logical development of two departments of general scientific theory. They occupy a fundamental position in general statistical theory. This in turn admits of application in most of the special sciences. But they also occupy a central position in the general theory of induction. They provide an essential premiss in any rational generalization. But it is important to distinguish two ways in which the principles of probability may be introduced. Let us suppose that we have examined 100,000 men and have found that in precisely 7 out of every 10 cases the average strength of grip is at least 4 kilograms greater with the right than with the left hand. We may raise two questions: (1) What is the probability that in any one of *these* men, selected at random, the two hands will differ in strength by this amount? (2) What is the probability that in the case of any man selected at random from *outside* this group the two hands will differ in this way? What, for example, is the probability that the right hand of Socrates was stronger than his left by this amount? It is obvious that the evidence supplied by the 100,000 examined cases is relevant to both these questions. It is obvious that the rational answer to this question must in some way be different in the light of the evidence from what it would be without it. But it is equally clear that the problems are of two different kinds. To the first we can give a simple and indisputable answer. It is that the probability is $\frac{7}{10}$. To the second question the answer is not so simple. It depends both on this evidence and upon whatever evidence there may be that the examined group can be taken as a fair representation of men generally or of some class including Socrates. But to the second part of this hypothesis no determinate probability can be assigned. Whilst, therefore, the theory of probability is applicable to both problems, it is applicable in very different ways. On this topic more will be heard in later chapters of this book.

BIBLIOGRAPHICAL NOTES.

CHAPTER X.

Together with the doctrine of the Categories, of the Predicables and of Definition, the theory of Classification and Division forms part of the classical doctrine of 'Terms'—Logic being subdivided into the three main departments of the theory of Terms, Judgments and of Reasonings. For a full discussion of these matters the student must be referred to the classical treatises mentioned in the Bibliographical Notes to Chapter V. The formal conditions of Division are treated most succinctly by **J. N. Keynes, Formal Logic,** Appendix A, and with important original features by **W. E. Johnson, Logic,** Part I., Chapter XI. In its bearings upon scientific method the subject receives full discussion in **Mill (Logic,** Book IV., Chapters VII. and VIII.), **Venn (Empirical Logic,** Chapters VII. and XIII), and **Jevons (Principles of Science,** Chapter XXX.).

On the statistical developments indicated in this chapter the standard treatise is **G. Udny Yule, Introduction to the Theory of Statistics; Karl Pearson, The Grammar of Science** (Chapter V.), may be consulted for a discussion of the subject in its bearings upon the general theory of scientific method. **L. L. Thurstone, The Fundamentals of Statistics,** provides an elementary introduction to the subject. Still more elementary is **W. P.** and **E. L. Elderton, Primer of Statistics.** On the general theory of Probability **J. M. Keynes, Treatise on Probability,** is the most authoritative modern work. (It contains an almost exhaustive bibliography.)

CHAPTER XI.

SCIENTIFIC THOUGHT AND SCIENTIFIC METHOD.

§ 1. The Nature of Science.

LOGIC is the study of the principles of valid inference. Science is one of the most important fields of inferential thought. It is therefore of interest to formulate the principles of inference by which scientific knowledge is obtained. Such a formulation would constitute a logic of scientific method.

Science rests upon a two-fold basis. There are, first of all, the *data* of science—the observed facts of nature. Secondly, there are the general principles upon which the scientist relies in passing beyond the data to the propositions of science which do not correspond to anything which is actually observed. The latter propositions include all scientific generalizations; since we do not actually observe a 'law of nature', we observe at most some of the instances of the law. They include propositions about unobservable entities such as atoms, electrons and sound waves, and propositions about hypothetical things. The general principles employed in passing from the data to the derivative propositions are not propositions of science,[1] but propositions of logic and mathematics used by the scientist in the construction of his science.

On a first glance at the problem it might be supposed that the empirical scientist and the logician co-operate in the extension of knowledge in the following way. The scientist ascertains the observable facts. The logician supplies him with the formal principles by means of which he can proceed from the data to the derivative propositions. These principles, it might further be supposed, are to be found somewhere in the deductive system of logical principles outlined in the preceding chapters of this book. But very little reflection is sufficient to show that such a view of the matter would be quite inadequate. For one thing it would not enable us to account for the process of scientific generalization. The data of science are *particular* observations. The scientist observes that such and such a particular piece of material substance has a certain property at a certain time and in a given situation. None of the logical principles with which we have so far been concerned will enable us to infer that this substance has the same or any other property at another time or in another setting. Still less can we infer anything about all material things of this kind. Given particular facts and all the laws of deductive logic we could only infer particular conclusions.

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[1] In this part of the inquiry we shall use the word 'science' only for 'empirical science' except where the more general signification is explicitly noted.

There are many other ways in which this simple view is very unsatisfactory. To obtain a more satisfactory view we must enlarge the scope of our inquiry. Instead of asking, simply: What principles of inference are required in science? we must inquire comprehensively into the nature of scientific thought as a whole. We must ask: What, more precisely, is the scientist trying to do? What general principles are involved in the methods he employs?

To the first of these questions various answers have from time to time been suggested. The scientist, it is sometimes said, is trying to *explain* the course of nature. This, however, is also sometimes denied. The scientist does not endeavour to explain, it is contended; he merely attempts to *describe*. Alternatively, it is suggested that he aims at *generalization* and *abstraction*, or that he attempts to *order* or *systematize* the facts. It has already been suggested in this book[1] that science is an attempt to provide a precise, complete, generalized and systematic account of things. But all these definitions are in a manner artificial, including the one favoured in our preliminary survey of the question. They are hopelessly inaccurate if they are intended as a genuine psychological account of actual scientific thought.

As a matter of psychological fact we can only say that the scientist is trying to do many different things. Sometimes, perhaps, he is trying to establish a general proposition; at other times his energies are bent towards the attainment of a more precise measurement of one particular thing—for example, the diameter of the earth. Sometimes he is trying to explain; sometimes he is trying to describe. As near as we can get to a general formulation we may say, perhaps, that he is always trying to solve a problem, to answer some specific question. A question, as we have seen, is an incompleted proposition. His answer completes the proposition. Completing it establishes a truth concerning the constitution of things. But in the course of solving problems he establishes general propositions, abstract formulations; and with the growth of knowledge there is a progressive revelation of system. We live, it seems, in a more or less orderly system of affairs. In consequence, the extension of our knowledge is an extension of knowledge of a system. If the facts were unsystematic we should still no doubt desire to know them. The scientist would not try to impose an order upon them if he did not find an order there. Things being what they are, the attempt to answer specific questions leads to greater completeness, greater precision, and system. It is in this sense that science 'aims' at knowledge of this kind.

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[1] Ch. I.

Given, then, that the scientist, whatever else he may be doing, is establishing the truth of propositions concerning the constitution of nature, what are the available methods and principles of procedure? The process, it would seem, has a dual character; it is a process of discovery and a process of proof. Under certain conditions the two operations coincide. The procedure which brings a proposition to mind simultaneously provides the evidence of its truth. This, however, is not always the case. Sometimes a proposition is very easily conceived, but extremely difficult to prove. At other times the difficulty lies in thinking of the required proposition. Once it is thought of evidence is soon adduced which shows that it is true. For the interval between conception and proof the proposition has the scientific status of an 'hypothesis'.

Let us suppose the scientist to be attempting to explain the movements of the planets.[1] The crude data are of an extremely complex and varied kind. They consist of observations of the positions of various planets at various times. These positions are measured from many different points of reference, varying with the position of the earth when the several measurements are made. The bare list of actual records taken would present only a somewhat chaotic catalogue. The very first step is to order the data. This might consist in reducing the measures to a common fixed point of reference. Sorting out the material in this way the observer may come to see that the various positions successively recorded concerning a given planet are all points on a certain ellipse. In this observation a variety of originally unorganized facts are 'ordered' by a single concept —a step of the greatest importance in scientific procedure. So important, and in some circumstances so difficult, may the operation be, that it has sometimes been regarded as the most important factor in scientific research. It has been called the 'colligation of facts'. The concept of an ellipse is the 'colligating concept' which, applied to the data, unifies them into a single whole.

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[1] Provisionally we will take 'explanation' to mean subsumption of a special law under wider laws.

So far, the operation is merely descriptive. The observed positions of the selected planet are described as occupying points on a certain ellipse. This description, however, suggests the possibility of various generalizations, viz.:

(i) Every position of the given planet between the time of the first observation and that of the last observation was on this ellipse.

(ii) Every position of this planet for some time before the first and for some time after the last observation is on this ellipse.

(iii) The positions of other planets are on some ellipse.

These generalizations, whilst suggested by the observed facts, are obviously not proved by these facts. So far they are merely hypotheses, the framing of which may serve to direct further observation with a view to establishing these propositions.

Given proof of the generalization that planets move in elliptical orbits the explanation of this remains to be found. This in turn involves two steps, the framing of explanatory hypotheses and the provision of proof that one of these hypotheses is correct. If, in general, this consists in the discovery of still more general laws from which the given law may be deduced, and if these more general laws admit of explanation in the same way, it is clear that the investigation must ultimately result in the formulation of an hierarchical system of generalizations of the kind outlined in Chapter I.

From such general considerations it is evident that the methods ' of science are broadly of two kinds, methods for the discovery of propositions, true or false, and methods of proving which of these propositions are true.

Concerning method in discovery there is little to be said, and what can be said is of slight practical importance. Strictly there is no method for discovery. We cannot frame promising hypotheses in accordance with a set of rules. The conception of a method implies a series of determinate steps which, taken in a certain order, give a certain required result. In this sense we may speak of a method for making an instrument, or for solving an equation. But from the nature of the case there can be no determinate series of steps which inevitably issue in a discovery. There is no method for discovering a needle in a haystack which would not break down if there is no needle there. A search can be more or less ' methodical '. We may subdivide the region to be explored into determinate sections each of which can be examined by a single act of perception. By examining each section in turn we can ensure that the whole field is explored, and that no section is examined more than once. There is, however, no infallible method which will prevent us from failing to see what is before our eyes. Moreover, even this amount of method is inapplicable to the search for a cause ; since we are not in general in a position to define in advance the field to be explored.

Methods of proof are in a somewhat different position. Here the question at issue is not that of finding facts which would prove a given proposition, but of defining the logical type of evidence which propositions of different forms require. We can broadly define the kind of premisses different types of scientific propositions would require. To formulate such principles is one of the most important functions of a logic of scientific method.

§ 2. The Data of Science.

Ignoring provisionally our primitive stock of self-evident beliefs, the data of science consist in the facts revealed to sense perception. These facts are the data both for the system of prescientific beliefs which we call common sense and for the system of scientific beliefs. Whether it be held that science is only a development and refinement of common sense or that it constitutes an opposed or divergent development of thought, it is clear that the two systems have a common origin and proceed for some distance, at any rate, along a common path. It will serve to bring into focus some of the main questions concerning scientific procedure if we attempt to define the nature of the data and the concepts employed or presupposed in the propositional formulation of the primary facts from which scientific thought proceeds.

The facts of perception are, as we have already noted, always particular facts. We perceive that a certain individual thing possesses a certain attribute, or that it stands in a certain relation to some other particular thing. Such facts are approximately expressed in statements of the form '*This is p*' or '*This is r to that*' where *p* stands for a sensible attribute such as *redness* or *roundness* and *r* stands for a sensible relation such as *adjoining* or *simultaneity*. Of course many such facts may be simultaneously apprehended. Nature appears to us as a multiplicity of particular things characterized by attributes, related, changing and interacting in various ways. But however many things may be perceived, we never perceive a generalization, or anything expressible in the form: Every A is B.

Although the presented data are approximately expressed in statements of the form '*This is p*' and '*This is r to that*' the degree of approximation is very crude. Such expressions fail to portray the fact with precision. Consider for example the statements:

'This is red'.
'This is near that'.

The quality actually perceived may be an absolutely specific shade of red; but the word 'red' does not express this specific shade. 'Red' is applicable to an immense number of shades. Similarly, in observing the distance between two objects we may apprehend a quite specific distance. The word 'near', on the other hand, is a purely relative term, and even when used with the greatest precision it is applicable to a large range of distances. There is an ineradicable element of indeterminateness in the words we employ to express what is observed.

Most qualities and relations it will be found belong to a certain series of qualities or relations of the same general kind. Thus we may distinguish the series of specific qualities consisting of all the shades of red, blue, yellow, green, the blacks and whites, and all the shades and tints resulting from the blending of these. This series we call the colours.

The words we employ to state the facts concerning the colours of things also form a series, but a series of a different kind. At one extreme there is the adjective 'coloured', which merely tells us that the thing to which it is applied has *some* colour without giving us any clue as to the position of this colour in the series. Adjectives such as red, green, blue, etc., restrict the possibilities to a certain region in the series, but they, too, are indeterminate. Special words such as scarlet, beige, ultramarine, introduce a further restriction; but there are no words by which we may express the specific shade which we observe in a given object. Similar considerations apply in the case of other qualities. Tastes and smells, for example, are even more difficult to express.

It is convenient to mark these distinctions between predications by certain technical terms. The general *sorts* of property may be described as '*determinables*'. Thus 'colour', 'shape', 'size' (spatial magnitude), etc., are determinables. Specific colours, shapes and sizes may be described as '*determinates*' under these determinables. We must also distinguish varying degrees of determinateness, ranging between the extremes of complete indeterminateness expressed by such predicates as 'coloured', 'shaped', 'extended', through the relatively determinate predications such as 'red', 'triangular', 'small', down to the absolutely determinate predications.

The first important distinction, therefore, to be noted between the statements expressive of our experience at the common-sense level and those of science is in respect of determinateness. For the subsequent operations of scientific thought it is necessary to formulate the facts in a manner which approximates more closely to the determinateness of the presented data. This is largely a matter of technical nomenclature and of specific devices for the purpose of measurement. For each determinable it is necessary to devise scales and units in terms of which to describe the precise position of any specific property or relation within the series comprised under the relevant determinables.

In the case of relational properties there arises a greater scope for indeterminateness and hence an increased complication in the procedure of specification. In predicating a relational property there may be greater or less determinateness both in respect of the relation involved and in respect of the character of the related terms implicated. It is obvious that a statement to the effect that something is *poisonous* conveys (i) some information concerning the intrinsic properties of the thing that poisons, (ii) some information concerning the changes undergone by something which is poisoned, and (iii) some information concerning the relation between these two things. Roughly to assert that X is poisonous is to assert that X is such that when brought into some spatial relation to something else, Y, there occurs sooner or later some change to the detriment of Y. Since it is not significant to assert of certain things, such as a mental state, that they are poisonous, the predication would seem to imply that X is a material substance, i.e. that it is characterized by some determinate property under the determinables involved in materiality. The changes in Y likewise must be definable under determinables of which death and organic injury are relatively determinate specifications. Some implication is further involved with regard to the spatial relations requisite for such interactions to occur and with regard to the temporal interval elapsing between the conjunction of the two things and the consequent effect. The process of specifying the determinate properties and relations involved in a particular case of poisoning constitutes the first step in a scientific investigation of the process in question. The scientist must commence by getting clear with regard to the facts; that is, he must ascertain the data with greater precision than is represented by the common-sense description of the case.

But even given clarity with regard to the data it is a far cry from these facts to the facts asserted in a textbook of science.

§ 3. Derivative Propositions.

Among the propositions which find a place in scientific systems and which seem to be far removed from those which merely embody the data of science two types call for special attention—those which assert uniformities or laws, and those which assert concerning particular things or sorts of things the possession of properties which are not directly revealed to perception.

Before raising questions concerning the evidence required to establish scientific propositions we must attempt to determine more precisely how such propositions differ from those that merely embody the data.

Uniformities.—The simple direct perception of a single property isolated from any context—if such a perception could occur—would not constitute a datum for any scientific purpose whatever. If, for example, we could experience in complete isolation a bare sensation of pain or the taste of lemonade the most that we could assert on the basis of any such experience would be that there is such a quality as pain or lemonade flavour, that something is painful, something is lemonade flavoured. Such propositions do not carry us far. Before science can begin we must at least experience a concomitance or sequence. Two or more properties must be manifested in some relation, as properties of the same thing or as separate manifestations in a given spatial or temporal relation. Given a pair of properties manifested as characterizing the same thing two possibilities are immediately opened up, (1) that this association is uniform, i.e. that in all other cases in which one property characterizes something the other property would characterize it too; (2) that the connection is fortuitous and irregular. The former possibility is the more interesting. It is the sort of thing the scientist endeavours to establish. Similar considerations apply to other relations, such as the relation of temporal sequence in the manifestation of properties when the properties in question do not characterize the same thing.

The simplest type of scientific 'law' is that expressed in a generalized statement of the form '*All X's are Y*'. Such statements, of course, are not distinctive of science. Among the propositions expressed in this form we may distinguish various degrees of abstractness. Every proposition, whether a generalization or not, involves abstraction. In asserting that '*This cow is cloven-hoofed*' we abstract the single property of being cloven-hoofed from the concatenation of other properties involved in the notion of a cow. So, too, in the generalization: *All cows are cloven-hoofed;* we isolate this property from the residual group of characteristics by reference to which a cow is defined. For some purposes it may be advisable to distinguish and explicitly symbolize all the properties involved. We then express the proposition in a statement of the form *Everything that is ABCDE is P.* Further inquiry may reveal that the truth of the generalization is not affected if we drop all reference to B, C, D and E and assert simply: *Every A is P,* where A and P are relatively simple properties. We thus obtain a more abstract law which concerns only the concomitance of two relatively simple properties both of which have been abstracted from the complex situation in which they are normally presented. It will be found that scientific generalizations generally differ from common-sense generalizations in attaining to a much higher degree of abstraction. Thus, comparing the two statements,

Water runs downhill,
Matter gravitates,

the former will be seen to be of the form which admits of explicit expression in the formula, *Every ABC . . . is PQR . . .*, where ABC . . . would symbolize the group of properties by which water is defined and PQR . . . the group of properties involved in the notion of 'running downhill'. The latter expression approximates to the form *Every A is P* where A and P are the relatively simple properties involved in the definition of matter and of gravitation. The more abstract law is reached by scientific analysis and empirical investigation which established the irrelevance of the properties BC . . . and QR. . . .

Nevertheless, many scientific generalizations necessarily involve reference to more than two properties. If, for example, we measure the current flowing through a wire, the electromotive force generated by the battery or other source of power, and the resistance of the wire we discover a law which admits of expression in the form:

> Every wire of resistance *r* through which a current determined by an electro-motive force, *e*, is passing, will manifest a determinate measure of current, *c*.

More generally, we reach what is known as Ohm's Law,

$$C = \frac{E}{R},$$

where C, E and R are variables standing for a whole range of measures of current, electro-motive force and resistance. This extension, however, involves a refinement of the process of generalization which will be considered in more detail later.[1]

Imperceptible Properties.—Scientific propositions are not all generalizations of the kind just examined. Even in a completed science we should find assertions concerning the properties of specific things and of specific kinds of things. In the textbooks we encounter statements such as the following: *Water boils at* 100° C. *The melting-point of sulphur is* 115° C. *The coefficient of expansion of platinum is* $\cdot 0837 \times 10^{-4}$ *per degree centigrade.* These statements are not merely expressions of what is directly observed. They do not merely express the data of science, they express something derivative from the data. They are not generalizations of the kind previously considered.

They are, however, generalizations. So far as they are generalizations concerning substances of a specific kind they are generalizations of the familiar sort. But there is a sense in which we can say of a particular sample of water that its boiling-point is 100° C., of a particular lump of sulphur that its melting-point is 115° C., and of a particular piece of platinum that its coefficient of expansion is $\cdot 0837 \times 10^{-4}$ per degree centigrade. On the other hand, in making such assertions we are saying something more than that these particular substances have been observed to undergo certain specific changes. This something more is also a generalization, viz. that the substance in question would always undergo these changes under certain conditions. Such simple statements therefore turn out to be more complex than at first sight appears. Statements of this kind concerning a substance of a certain *sort* contain a double generalization—a generalization to the effect that a certain generalization can be made with regard to every sample of a substance of the given kind.

More complicated examples of this kind of generalization are provided by the more precise concepts employed by psychologists in the specification of abilities and traits of character.

There is an obvious contrast between statements to the effect that S, a given individual, has solved a certain problem, or is experiencing a certain emotion and the, in some way related, statement that S is intelligent or emotional. At first sight it might seem that this contrast is one of material content, that it is purely a psychological difference with which the logician as such is not concerned. But very little analysis is sufficient to show that the assertions differ in logical form, that the former directly express data and the latter do not, and that something of the nature of a logical principle is required to explain how the latter propositions can be derived from the former. The difference, in fact, is precisely analogous to that between propositions embodying the data of physics and propositions concerning coefficients and the 'specific properties' of different kinds of material substance.

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[1] In Ch. XVI.

The psychologist, like the physical scientist, starts from the observation of particular facts. In his case the facts are particular acts of thought, of recall, particular states of emotional excitement, etc. Most psychologists would include among their data, and some would give priority to, particular facts concerning behaviour and bodily expression. But in every case the facts would be admitted to be of the form

S is P at T.

Where S is an individual, P a property manifested in consciousness or behaviour, and T a certain time. Formally these are facts of the same kind as those of the physicists—S might be a piece of sulphur, P a change from the solid to the liquid state and T a time.

Having observed that S is P at T, the simplest operation is to proceed to the bare statement, S is P, dropping all reference to the time. What would such an expression signify? It is, of course, an indeterminate assertion to which no precise significance can be attached. As commonly employed, however, the statement expresses a simple generalization: S is P always. Such an assertion, however, would be almost invariably false. It is obviously false so to generalize the observation S has solved the problem, S is experiencing an emotion. It is impossible even to generalize in this way statements such as: This rose is red, This piece of sulphur is yellow.

Legitimate generalizations are almost invariably of a more restricted kind. When from the observation that certain positions of a given planet were on an ellipse Kepler implicitly formulated the generalization *This planet has an elliptical orbit,* we must understand the tacit qualification to be implied ' throughout the present epoch ' where the expression connotes a restriction, though an indeterminate restriction, to a period of time with which the investigation is concerned. Kepler was concerned, that is to say, with the problem of explaining the movements of the planets during a certain period of cosmic history in which the solar system manifests a more or less stable structure. Had he been concerned with a longer period embracing the origin and the possible extinction of this system a different generalization would have been required. In generalizations of this kind the nature of the restriction is more or less adequately suggested by the context of the assertion. It is therefore not usually expressed.

The second type of restriction is one which requires explicit formulation. This is the case in which the generalization depends upon some further condition. It may be expressed in the form *S is P if Q*. Most generalizations concerning an individual thing are of this form. This sample of sulphur melts at 115° C. means : This sample of sulphur will change from the solid to the molten state whenever its temperature rises to 115° C. In this formula, too, we obtain a first approximation to an analysis of the propositions ' S is intelligent ', ' S is irritable '. These assertions may be interpreted as meaning: Whenever S is confronted with a problem of the kind *p* he will discover the solution ; Whenever S is placed in a provoking situation he will manifest irritation. These, however, we shall find to be only first approximations. Such propositions are not the only kind to be derived from the data. Two other cases requiring examination are frequently found in science : statements concerning *potentialities* and statements concerning *tendencies.*

The conception of a *potentiality* is illustrated by statements concerning ' abilities ', ' faculties ', ' powers ' and ' capacities ', in their applications both to mental and to physical processes. They find expression in the formula : *S can do so and so, S can be P.* This is the primary significance of propositions such as *So and so has a good memory, Gold has the property of malleability to a higher degree than other metals.*

The force of such propositions is not adequately expressed in one of the forms : *Every S is P* or *S is always P.* The primary reference is to a *limit* in a series of specific properties, p_1, p_2

. . . p_n. Let us suppose that the only manifestation of memory lies in the recall of poems of varying numbers of lines, and that the only difference between one individual and another in respect of memorization lies in the number of lines he can accurately recall after a standard number of readings. S_1, let us say, recalls n lines; S_2 recalls $n + m$ lines. We should then say that S_2 has the better memory. We define memory capacity by reference to the maximum number of lines recalled under standard conditions. S_2 in recalling $n + m$ lines recalls also $1, 2, 3, \ldots n, n + 1 \ldots n + m$ lines, i.e. if he 'can' recall $n + m$ lines he can also recall any smaller number of lines. But when we assert without qualification that he can recall $n + m$ lines it is understood that this is the maximum number, the limiting number. In point of fact, of course, memory is not restricted to a given type of material. When we assert that so and so has a good memory we imply that he can remember all sorts of things with efficiency. Defining his capacity not in terms of an absolute scale, but in terms of the average ability of men, the statement suggests that there is a certain limit above the average to which the individual in question can remember various sorts of material. Thus, although the primary significance of the assertion that certain individuals have a certain ability lies in its reference to a limit it will be seen that it involves no less than three forms of generalization. Let the proposition be *Europeans have good memories.* The data, let us suppose, are provided by the application of memory tests to a group of Europeans. The transition to the derivative assertion involves:

(*a*) The generalization that the individuals who performed the tests with a degree of efficiency, e, at the time of the test would perform this test with equal efficiency at other times.

(*b*) The generalization that other individuals of the same class (i.e. Europeans) would perform the test with equal efficiency.

(*c*) The generalization that the individuals referred to would perform operations of memory other than those comprised within the test with equal efficiency.

In other words, it is presupposed that the acts of memory involved in the test are a fair sample of (*a*) the performance of the same individuals at other times, (*b*) the performance of other individuals of the same class, and (*c*) the performance of these individuals in allied but different operations of memory.

Similar considerations apply to physical statements such, for example, as the statement that gold is malleable. The data consist in observations relating to substances being beaten into thin sheets, and in the observation that some of these substances maintain their continuity under this process for a longer time than others. Thus it is found that a given ounce of gold may be beaten out to yield two to three hundred square feet of leaf, a figure much higher than has been obtained with other metals. We thus obtain a series of gradually decreasing dimensions to which the thickness of various samples of metal may be reduced before rupture occurs. Defining this limit would yield a 'coefficient of malleability'. Such a coefficient would imply at least two generalizations:

(*a*) That if the same ounce of gold were repeatedly beaten out and melted down again on each occasion of beating out we should reach the same limit.

(*b*) That this applies to every sample of gold.

If, moreover, we extended the notion of malleability to cover other processes (such as drawing out into thin wires) a third form of generalization would be involved analogous to the generalization of the range of operations comprised in our previous example under the notion of 'memory'.

Among the derivative propositions of science we may note, finally, those expressed by statements of the form: *S has a tendency to be P.* This is a type of assertion common both to science and to everyday thought. A spring has a tendency to unbend; mercuric fulminate has a tendency to explode; puppies have a tendency to destroy slippers.

We must first distinguish senses of the word 'tendency' according as they involve notions specific to some restricted science or only purely logical notions. As an example of the first sense we may notice the psychological usage in which a 'tendency' is a 'felt tendency'. To assert that some one has a tendency to laugh on unsuitable occasions may mean simply that he experiences a desire or an impulse to laugh on such occasions. This introduces conceptions specific to psychology. A more purely logical sense of the term is illustrated by statistical statements. In throwing a die we may observe that it falls, with the 'six' uppermost, not invariably but much more frequently than the laws of probability would lead us to expect. We express this fact by saying that the die manifests a special tendency to fall in this way.

In this usage '*S has a tendency to be P*' expresses an approximation to a generalization, something more than '*Some S's are P*', something less than '*Every S is P*'. It expresses the relation between two properties which are neither wholly independent nor invariably conjoined. The investigation of these relations belongs as we saw in Chapter X. to the statistical theory of *Association.*

The logical distinction between potentialities and tendencies will be found to underly the distinction in psychology between abilities and traits of character. Neither are directly observable features in an individual. Both require to be derived by some logical operation. To say that an individual, S, has an *ability* to perform the operation P, does not imply anything with regard to the frequency with which he does in fact perform this operation. To assert that he has a *tendency* to perform this operation carries the implication that he performs it with abnormal frequency.[1] This defines a trait in his character. The distinction is well exemplified in the difference between the statements '*S has great imagination*' and '*S is imaginative*'. The former statement would be justified by evidence of a single act of marked imaginative construction, such as, for example, the production of a single work of art, even though S in every other respect and at all other times exhibits no indications of imaginative activity. The assertion that S is imaginative, on the other hand, presupposes a relatively high frequency of imaginative acts. The former statement predicates an ability, the latter a tendency or a trait of character.

Though perhaps more clearly marked in the psychological applications the distinction holds throughout the whole field of scientific inquiry. Thus to revert to our physical example, the malleability of gold is a potentiality ; its tendency to assume, in its crystallic states, the form of an octahedron belongs to its 'character'.

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[1] There are, of course, other senses of the word 'tendency'. 'S has a tendency to phthisis' asserts that S would more readily develop this disease than anyone of normal constitution, that the circumstances requisite to produce it are in his case simpler than in the case of a normal person.

BIBLIOGRAPHICAL NOTES.

CHAPTER XI.

There is a very extensive literature relating to scientific method.

Although a study of **Francis Bacon** and his precursors would provide an appropriate historical introduction to the subject, the student will find in **Mill's System of Logic** the most convenient point of departure. Other standard works are : **Sigwart, Logic ; Karl Pearson, The Gramma of Science ; Venn, Empirical Logic,** and **Jevons, The Principles of Science**

Among the many works by contemporary writers the following ma be mentioned : **A. D. Ritchie, Scientific Method ; N. R. Campbell, What is Science ?** (an elementary introduction) and the same author's more technical work, **Physics, the Elements.**

The student will appreciate more fully the problems of this branch of Logic it his studies are related to some special branch of science with which he has first-hand acquaintance. Methodological discussions pertaining to special sciences occupy a prominent position in the philosophical journals, notably **Mind, The Monist** and the **Journal of Philosophy** (previously **The Journal of Philosophy, Psychology and Scientific Methods). The Proceedings of the Aristotelian Society** should also be consulted.

It is important throughout these studies to preserve clearly the distinction between the theory of scientific *proof* and the theory of scientific *discovery*. It is important for an understanding of the controversy between Mill and Whewell (cf. **Johnson, Logic,** II., IX., § 2), and for disentangling many issues in more recent contributions to the subject. The latter topic—the theory of discovery—is primarily a question of Psychology. As such it is treated in an elementary way in Chapter IV. of the author's **Psychology of Study.** Reference may also be made to the more general psychological studies of the process of thought mentioned in the Bibliographical notes to Chapter II.

The distinctions drawn in § 3 of the text between different types of derivative propositions in science—explicit generalization, the attribution of potentialities, tendencies and so forth—has also been discussed by the author in an article entitled '**Faculties and Instincts**', **Mind,** Vol. XL., Jan., 1931.

It may, perhaps, be noted as one of the more important omissions in this chapter that no reference is made to the fundamental *concepts* of scientific thought. Science employs not only a certain method and propositions of certain logical forms, it also employs certain basic concepts—Space, Time, Substance, Causality and so on. In the biological sciences, it is often suggested, other basic concepts are required—the concept of an 'organic whole', and concepts of a 'teleological' character. The student will not be able to progress very far in the study of scientific method without turning his attention to these matters. In the text, however, an attempt has been made to restrict discussion to the more purely logical topics as contrasted with those that belong to what might be described as the 'philosophy of science'.

CHAPTER XII.

TYPES OF INDUCTION.

§ 1. Induction and Generalization.

THE preceding examination of the general nature of the data of science and of its derivative propositions has thrown into prominence the central position of problems of induction. Science begins with particular facts. It ends with generalizations. Whatever assurance we may have in the generalizations depends upon the evidence of the particular facts. The question thus arises: By what logical operations can we reach generalizations from data concerning particular cases?

Induction may be defined as a form of inference in which, given instantial premisses (i.e. premisses concerning 'instances' or particular cases), we draw a conclusion involving some measure of generalization of these premisses. It will be noted that this definition does not insist upon any absolute or complete generalization of the premisses. It is desirable to employ the term 'induction' so as to cover partial or restricted generalization or any extension beyond the instances contained in the premisses. Thus we must include inferences such as the following: '*Since* 50 *per cent. of the criminals convicted in the first decade of the present century were mentally abnormal, it is probable that* 50 *per cent. of those convicted in the second decade were also abnormal*'. We should also recognize as an inductive inference the following: *Since every examined P was found to be Q, probably the next P we encounter will be Q*'. We may also include under induction arguments from analogy: '*Since Mars resembles the Earth in many important respects, and the Earth is inhabited, Mars is probably inhabited*'.

It will also be observed that, as here defined, induction may be found to comprise both demonstrative and problematic forms of inference. All that is necessary is that there should be an instantial premiss and some measure of generalization. There may be other premisses which, combined with the instantial premisses, render the conclusion strictly demonstrative.

It will be convenient to commence with cases of complete generalization, and to raise the very wide question: How are general propositions proved? By progressive elimination of the less important cases we shall bring to light the distinctive group of problems with which the logic of scientific method is concerned.

One case, in particular, may be quickly disposed of. Some universal propositions are established by pure deduction. This is illustrated by syllogistic arguments. But deductive arguments never constitute the ultimate ground, or an ultimate explanation of our acceptance of the conclusion. This is obvious when we consider the fact that a universal conclusion in a deductive argument requires universal premisses. If the question be pressed: What are the grounds for the acceptance of the premisses? we must either put them forward as general propositions accepted without evidence or refer to some form of inference other than deduction.

Another non-inductive case requires to be noticed. There is a proposition of considerable importance in mathematics which can be expressed in the following terms: *If any property belongs to the number* 0, *and is such that if it belongs to any number then it belongs to its successor; then it belongs to all numbers.* That is to say, if we can prove of any property,

(*a*) that it belongs to 0;
(*b*) that if it belongs to n it belongs to $n + 1$;

then we may assert that it belongs to any finite number.

This is the so-called principle of mathematical induction. It is not easy to see how we know this general proposition to be true. The difficulty disappears, however, if we employ this proposition to *define* the series of natural numbers. That is to say it becomes true because if we found anything of which it is not true we should refuse to include it in the series of finite numbers. We know this proposition as we know that in every circle all points in the circumference are equidistant from the centre. This follows, as we say, from the definition. Obviously, in such cases there is no problem of explaining how we know the general proposition to be true. The general proposition is a tautology.[1]

We may, therefore, dismiss as irrelevant to our problem all universal propositions established by purely deductive methods, and those that merely 'follow from definitions'. The cases that remain to be considered may be distinguished under four heads: Perfect or Summary Induction, Intuitive Induction, Demonstrative Induction and Problematic Induction. The first two of these are of some historical interest, having occupied a prominent position in inductive theory since the time of Aristotle. The third introduces general questions concerning the interconnection of deductive and inductive methods, but in the fourth alone do we approach anything like an ultimate account of the logical basis of typical scientific generalizations.

§ 2. Perfect or Summary Induction.

Murder, let us suppose, has been committed at the Grange. The detective in charge of the case notes that five persons were in the house at the time, A, B, C, D, and E. Cross-examining each in turn he finds that each has a perfect alibi. He concludes that all the members of the household have alibis, and proceeds to look for the criminal elsewhere.

This somewhat trivial operation was one of the forms of induction recognized in Aristotelian Logic. It has been commonly described as 'perfect' induction on the ground that the universal conclusion is obtained on the basis of a complete enumeration of the particular cases comprised under the universal conclusion. What is asserted with regard to *every* case in the conclusion has been ascertained for *each* case individually in the premisses.

The process may be represented symbolically in the form:

$S_1, S_2, S_3, \ldots S_n$ are P.
$S_1, S_2, S_3, \ldots S_n$ are all the S's there are.
$\therefore$ Every S is P.

This is called 'the inductive syllogism'. It differs from the ordinary categorical syllogism in that the premisses have an enumeration of items in the place of the ordinary subject and in the fact that the minor premiss has a quantified predicate. It is this latter feature that makes it possible for a universal conclusion to be drawn from what appears to be an argument in Figure III.

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[1] Other views may be taken concerning mathematical induction, the most plausible of which is that it belongs to the type to be described as Intuitive induction.

The operation involved in summary induction is trivial but not altogether unimportant. It is not unimportant that the detective should be able to restrict the field of his investigation by the elimination of a determinate group of original suspects. A similar process occurs in one stage in the inductive process in which an inference is made from a proposition concerning all examined cases of a certain kind to a conclusion concerning all cases whatever. This process may be represented as follows :

S_1, S_2, . . . S_n are P.
S_1, S_2, . . . S_n are all the examined cases of S.
∴ All examined cases of S are P.
∴ All cases of S are (probably) P.

The process of summary induction also occurs at another stage of induction. This is illustrated by the following case:

The cow, the sheep, the deer, etc., are ruminants.
The cow, the sheep, the deer, etc., are horned animals (and are *all* the horned animals).
∴ All horned animals ruminate.

This in point of fact was the kind of case which Aristotle chiefly had in mind. It differs from the first example in that the items enumerated are not individuals but species. Consequently the process is one in which a generalization is proved concerning a genus by an enumeration of generalizations concerning the species comprised within the genus. The same principle is exemplified in the argument:

Every parabola, every ellipse and every hyperbola meets a straight line at less than three points.
The parabola, the ellipse and the hyperbola are the conic sections.
∴ All conic sections meet a straight line at less than three points.

Here, too, the premisses are not concerning individuals but classes. Having, by some process other than summary induction, proved something concerning each of a number of classes we have established by summary induction that the members of some other class (co-extensive with the enumerated classes) possess this property.

In the foregoing cases the conclusion is in every case a universal proposition restricted to a relatively small finite number of cases, a finite number of individuals or species. The question arises whether summary induction admits under any circumstances of an unlimited universal conclusion. That it does admit of this kind of conclusion is contended by Johnson,[1] but not apparently by any other logician. The issue depends upon whether under any circumstance we are able to 'examine' an infinite number of instances.

The case cited by Johnson is that involved in a 'geometrical' as distinct from an 'analytical' proof of a geometrical theorem.[2] His example is the proposition '*Two straight lines terminating at the same point cannot intersect at another point*'. Clearly, this proposition cannot be known to be true by an examination in perception of every case covered by the proposition. The truth of this proposition, it is argued by Johnson, is established by summary induction the instances of which are provided by imagery. There are two infinite series of cases comprised within the general assertion, (*a*) the series obtained by revolving one of the lines through 360° from any position to which it returns; and (*b*) the series obtained by extending the lines in both directions. There is, of course, an infinite number of extensions possible for any case under (*a*).

It is only if such a process of imagery is possible, says Johnson, 'that we can say that the axiom in its universality presents to us a self-evident truth'. But as a point of psychological fact it is clearly not the case that such a process of imagery is possible at all. Granting the possibilities of visualizing the series of cases comprised with the series (*a*), it remains wholly impossible to visualize the infinite series of extensions required under (*b*). It would seem rather that the knowledge of the proposition in question is obtained by a species of 'intuitive induction'; though, as Johnson points out, the case differs in one important respect from those commonly brought under the latter principle. It requires consideration of a larger number of cases.

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[1] 'Logic', Pt. II., Ch. IX.

[2] An 'analytical' proof is one of the same logical character as an algebraical proof, not dependent in any way upon the employment of a figure. A geometrical proof depends upon the perception of properties specific to *spatial* relations and is thus mediated by reference to a figure.

§ 3. Intuitive Induction.

A second process through which we may be led to assert a universal proposition in consequence of the examination of particular cases is that commonly described as 'seeing the general rule in the particular case'.[1] We perceive, for example, that a given closed figure is composed of three straight lines and that it has three angles. We can immediately generalize this, and assert that any other closed figure composed of three straight lines will have three angles. The significance of this case is appreciated when we contrast it with the following. We perceive also that the given figure has one right angle. Here, however, we cannot proceed to the assertion: Every figure composed of three straight lines will contain one right angle.

Deferring analysis of the nature of this operation we may notice that it is the principal source of our knowledge of 'axiomatic principles'. It is the source of what in Chapter I. we described as our primitive stock of self-evident beliefs.

Such axiomatic or self-evident general propositions fall broadly into two groups. Some relate to the general relations between special kinds of properties such as colours, tones and so on, and special relations such as spatial and temporal relations; others relate to much more general properties and relations, such as those involved in mathematics and logic.

Thus we see that

$$\begin{aligned}&10 \text{ times } (4 \text{ days} + 6 \text{ days})\\ &= (10 \text{ times } 4 \text{ days}) + (10 \text{ times } 6 \text{ days});\end{aligned}$$

also that

$$\begin{aligned}&3 \text{ times } (7 \text{ yards} + 5 \text{ yards})\\ &= (3 \text{ times } 7 \text{ yards}) + (3 \text{ times } 5 \text{ yards});\end{aligned}$$

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[1] This is another of the processes of induction recognized by Aristotle though he would seem to have included within its scope many propositions which would now be generally agreed to depend upon the principle of 'problematic' induction. A more restricted form of the doctrine of 'intuitive induction' is that given by Johnson, 'Logic', Bk. II., Ch. VIII.

and so proceed to the understanding of the general truth that

$$x \text{ times } (y + z) = xy + xz,$$

where y and z stand for homogeneous quantities.

In grasping this general proposition we have come to apprehend the Distributive Law in its ordinary algebraical form. By a precisely similar process we may come to see the validity of the corresponding logical principle. If we know concerning two doctors in partnership, A and B, and their assistant, C, that during consulting hours

C is present and either *A is present* or *B is present*,

we then know

Either *A and C are present* or *B and C are present.*

So, too, if we know that a certain announcement appeared in the agony columns of *The Times* on 1st January, and that this announcement also appeared in *The Times* either on 1st February or on 1st March, we then know that it appeared either on both 1st January and 1st February or on both 1st January and 1st March.

We see that the validity of the first of these inferences is wholly independent of any peculiarity in the behaviour of the doctors and that the second is wholly independent of the characteristic merits of *The Times* as a medium of publication. The inferences, we see, would hold whatever propositions we employed. Hence, we see, the general principle expressed by the formula

$$p(q \vee r) \equiv p \,.\, q \vee p \,.\, r,$$

where p, q and r may stand for any proposition whatever.

By what process, precisely, are these generalizations effected? In general terms, the transition is of the following kind. In observation we are confronted with particulars characterized by two or more properties, or with particulars characterized by properties and standing in relation to other particulars. Thus the data of perception may be expressed in propositions of the following forms: *This is both P and Q*, *This is P but not Q*, *This is P and it stands in relation R to that*, and so on.

In all such cases we can *conceive* a general rule of which the given datum is an instance. Having perceived that *this is both P and Q* we can conceive the possibility that *Everything that is P is Q;* and having perceived that *this is P and stands in the* *relation R to that* we can conceive the general proposition: *Everything that is P stands in the relation R to something.* But though we can always conceive such general propositions we usually have no reason to suppose that they are true. Having perceived that *A's car is red and noisy* we can conceive the proposition *Everything red is noisy.* This generalization would be false.

When, on the other hand, we observe that *A's car is wholly red* and that *it is not wholly green* we can proceed to the general proposition *Whatever is wholly red is not wholly green* with conviction of the truth of this generalization.

We may, in fact, proceed further to the more general proposition:

> *Whatever is wholly of one specific colour is not wholly of any other specific shade of colour.*

Still further we see

> *Whatever is characterized by one absolutely determinate property under a given determinable is not (in the same sense) characterized by any other determinate property under the same determinable.*

This latter principle requires rather more careful formulation, but it is easy to see that in some sense it is true that codeterminates under the same determinable are mutually exclusive. These three propositions, involving progressive generalization, show that no hard and fast line can be drawn between the case of material principles and that of formal principles. The first generalization relates only to specific kinds of properties: the third approximates to the kind of principle involved in Logic. The cases agree in one essential respect. The conditions which suffice for the proposition to be conceived also suffice for us to know that it is true. This is characteristic of all cases of intuitive induction.

The attempt to base our knowledge of general logical principles upon a special kind of induction would appear to offer a via media between the two extreme views involved in the historical controversy between rationalism and empiricism. According to rationalism, in the simplest form of this doctrine, our knowledge of axioms is innate, a priori or independent of experience. The opposed view, in its extreme form, was that our knowledge of these principles is obtained in precisely the same way as that of empirical laws of nature. The doctrine of intuitive induction bases this knowledge upon induction but concedes to rationalism that the induction is at least of a special kind.

It is questionable whether much is saved for empiricism by this expedient. The operation is directly or indirectly dependent upon knowledge of some instances but the propositions asserting that there are such instances do not seem to function as premisses in an argument. The perception of the instances performs the purely psychological function of enabling us to *conceive* the universal proposition. They supply the requisite components of this proposition, but they contribute nothing to its proof. If it be possible to imagine coloured objects or geometrical figures independently of sense perception, as it quite well may be, we can know certain generalizations about these objects without having any reason to suppose that there are any instances whatever conforming to these particular generalizations. Perception of actual ellipses for example might enable us to generalize concerning perfect circles even though a perfect circle had never been perceived.

So far as any positive account can be given of the process, we fail to find in it any clear exemplification of the distinction between premiss and conclusion by reference to which inference was defined. If a distinction is to be drawn between the data and what is derived from the data the relation is one that is very different from the relation of implication. 'This is red, and this is not green' does not *imply* 'Whatever is red is not green'. The most we can say is that the natures of '*red*' and '*green*' are such that they cannot both simultaneously characterize the same surface.

We may, perhaps, say that the evidence for the general proposition resides in the nature of its terms. It is in this sense that these propositions are *self*-evident. The evidence does not reside in anything outside the proposition. But for this reason, too, we cannot describe the process as one of 'inference', as the latter is ordinarily defined.

This analysis enables us to define with rather more precision the number of instances required in order to apprehend the truth of the generalization. This number will be simply the number of instances required in order to *conceive* the proposition, i.e. to entertain it in a purely hypothetical way. We have seen that in some cases it may be possible to conceive a general proposition concerning a certain specific property without knowing any instances whatever. So far as instances are required it would seem that the number varies with the generality of the propositions. In the case of the series of propositions of increasing generality derived from the observation that what is red is not green an increasing number of cases is required. For the first number of the series it would seem to be sufficient that we should have had experience of at least one red object and at least one green object. In order to know that every specific colour is incompatible with every other we must have had a larger number of instances- a number large enough to enable us to conceive the genus or determinable. For the most general proposition of all we must have had corresponding experiences relating to properties under more than one determinable.

The fact that more than one instance is sometimes necessary for intuitive induction tends to obliterate the distinction between the latter and the so-called process of 'geometrical induction'. In our apprehension of the truth of the proposition that two straight lines meeting at one point do not meet at any other, it is necessary to consider more than one pair of points on these lines other than the points of intersection. It is not necessary, however, to consider an infinite number of such pairs of points. From a small finite selection we are able to intuit 'the principle of divergence' with its implication that however far extended the two lines will not meet.

§ 4. Demonstrative Induction.

It is characteristic of certain phases of scientific procedure, more especially in the advanced sciences, that a single observation may provide the basis of an extensive generalization. Thus, in 1894 Lord Rayleigh and Sir William Ramsay in discovering a new element, *argon*, were presented with the problem of determining its properties. Rayleigh weighed a sample consisting of 163 c.c. of the gas and found its density to be 19·941 on a scale in which oxygen is represented by 16·0. This was accepted as a measure of the density of argon generally. Subsequent measures were in fact made, but this was merely for the purpose of making the measure absolutely precise. Rayleigh's single measure would have been accepted as correct to at least one decimal place.

Similar cases occur in everyday experience. A gardener observes that a plant, languishing in a shady position, flourishes when transplanted to a sunny spot. He generalizes that the species requires direct sunshine. On the other hand, he would hesitate to generalize from the fact that in one year it produces twelve blooms. There is no reason to suppose that it will always produce precisely this number of blooms.

It is easy to state in general terms the difference between the two cases. We should say with regard to the density of argon, that the density of an element is one of its constant properties. We have found this to be the case with other elements, and expect that argon will conform to the rule. Similarly with the gardener's inference. We find that members of a given species of plant are relatively uniform in respect of their general conditions of growth. But very few species produce a uniform number of blooms each year.

The type of inference here employed has been distinguished by Johnson as 'demonstrative induction'. It is inductive since it depends upon an instantial premiss. It is demonstrative, not because the conclusion is certain, but because whatever degree of probability the premisses possess is transferred without loss to the conclusion. Formally, the process can be stated as a particular case of the 'hypothetical syllogism'. The assertion '*Elements are constant in respect of density*' implies the narrower generalization:

If any specimen of an element, E, has a density, d, then all specimens of E have a density d.

This serves as a major premiss. The premiss,

This specimen of E has density d,

is the instantial minor premiss supplied by observation. Affirming the antecedent justifies the affirmation of the consequent:

All specimens of E will have a density d.

More generally, the formula of demonstrative induction may be stated:

Major. *If any S is P Every S is P.*
Minor. *This S is P.*
Conclusion. *Every S is P.*

Of course, the major premiss does not normally present itself in this form, but in a slightly more complex form :

If any S is x then Every S is x,

where *x* is a variable representing indifferently any values within an assignable range. We may thus express the principle of demonstrative induction in the form :

If any S is x then Every S is x.
This S is P.
$\therefore$ *Every S is P.*

This form brings out the characteristic of the type of inference which Johnson expressed by saying that conclusion is a *specification* of the major premiss and a *generalization* of the minor premiss. The form of the major premiss is characteristic of all cases in which generalization is possible on the basis of a single instance. The inference is one from 'any' to 'every'.

Alternatively, we may represent the major premiss as consisting of a *conjunction* of implicative propositions with a common subject and a series of predicates corresponding to the total range of determinate properties under an assigned determinable. Thus

Major premiss.	*If any S is* P_1 *All S's are* P_1
	If any S is P_2 *All S's are* P_2,
	. . . . .
	If any S is P_z *All S's are* P .
Minor premiss.	*This S is (say)* P_n
Conclusion.	*Every S is* P_n.

The most important applications of the principle of demonstrative induction are those in which it is possible to reduce a certain part of inductive investigation to a set of formal 'methods' of the type suggested by Bacon and Mill. The development of this subject belongs to Chapter XV.

The principle of demonstrative induction is also employed whenever it is possible to apply *tests*. A test is applied whenever we ascertain whether something has a property *p* in order to determine whether it has a property *q*. The validity of the test depends upon the truth of the proposition *If S is p it is q*. The test observation, however, merely determines that the object in question has the significant property at the time of observation, i.e. it yields the proposition *S is p at* t_1. Now, if this merely enabled us to infer that *S is q at* t_1 the test, though theoretically valid, would be practically of little value. If the test is to be of practical value we require to know the proposition *If S is p at* t_1 *then it is q at all times*, or at least for some time more extensive than is involved in the application of the test.

The simplest case is that in which we employ the possession of a property at one time as the sign of the possession of the same property over a longer time. For example, we may test the reliability of a watch by comparing the movement of the hands with a standard throughout the course of twenty-four hours. If it is running accurately for this period we infer that it will be equally accurate for a much longer period. If it has gained or lost a minute during the test it will gain or lose an equal amount every day. The major premiss of the test is thus:

If S is p_1 *from* t_1 *to* t_2 *it will be* p_1 *from* t_2 *to* t_n,
If S is p_2 *from* t_1 *to* t_2 *it will be* p_2 *from* t_2 *to* t_n,
If S is p_3 . . . *etc.*

The period t_2 to t_n may be more or less indeterminate, but must be of sufficient length to make the test worth applying. When there is a reason to doubt this premiss the test is not applicable. This would be the case, for example, if the watch suffered from a defect which was consistent with its keeping good time for several days on end, but which rendered it liable to lose or gain at intervals.

In the same way we may notice that an intelligence test presupposes the prior induction that intelligence is an individual constant, that whatever degree of intelligence an individual possesses on one day he is likely to possess over a considerable period.

Lack of assurance on this point constitutes an inherent difficulty in providing tests of many other mental qualities. A test of honesty, for example, would be proved to be theoretically impossible if it were shown that this quality is one which varies from time to time in the same individual.

It is clear from such general considerations that tests are of two main types: those which depend upon *samples* and those which depend upon *symptoms*. Thus a doctor might diagnose the presence of a certain disease either by the discovery of specific germs or of organic states actually constituent in the disease, or by the observation of 'symptoms' which are invariably concomitant with the disease. In a precisely analogous way a psychologist might ascertain that an individual possesses a certain ability by tests which depend upon the actual exercise of this ability, or by tests which invoke some performance dependent upon another ability invariably associated with the one under consideration. The formula of a test dependent upon 'symptomatic' manifestations is the following:

If S is p at any time S is q at all times

(or within an assignable period with which the test is concerned). The possibility of any such test depends upon two prior generalizations: (*a*) That if *q* is present at any time it is present at all times, i.e. that *q* is a constant property; and (*b*) that if *p* is present at any time *q* is present at the same time. Tests involving only the principle of sampling involve the simplified formula obtained by substituting *p* in place of *q* in the hypothetical major premiss. It is thus

If S is p at any time S is p at all times.

This involves only the simpler generalization that *p* is a constant property.

§ 5. Problematic Induction.

That parrots can talk, that bad teeth are associated with deficiency in vitamin D, that the resistance of nearly all metals is reduced to zero at a temperature of — 273° C. are propositions that cannot be completely established by the processes of inference so far considered. They are not proved by pure deduction, or by demonstrative induction. They do not follow from definitions. They are not established by an examination of every instance. They are not intuitive generalizations.

Such propositions, moreover, are typical of the vast majority of the propositions of science. They are typical 'empirical generalizations'. The field of empirical generalization is the field of problematic induction. Problematic induction comprises a residual group of processes excluded from the types of induction so far considered. The positive characteristics common to this group are (*a*) that they are inductions in that generalization depends upon the examination of instances, and (*b*) that they are problematic in as much as they yield conclusions which are less certain than the premisses.

Traditionally, the problem of scientific induction has been formulated by logicians as primarily, if not exclusively, the problem of complete generalization. Analysis has been largely restricted to the cases in which an inference is made from a premiss of the form *Every examined P is Q* to a conclusion of the form *Every P is Q.* In this respect Mill's statement is representative. 'Induction', he says, 'may be summarily defined as Generalization from Experience'. It consists in 'inferring from some individual instance in which a phenomenon is observed to occur, that it occurs in all instances of a certain class'.[1] Earlier he states, 'We shall fall into no error if in treating of induction we limit our attention to the establishment of general propositions. The principles and rules of Induction, as directed to this end, are the principles and rules of all Induction; and the logic of science is the Universal Logic applicable to all inquiries in which man can engage'.[2]

This is possibly the case, but it is undesirable to assume it in the present unsettled state of inductive theory. It is better to view the matter more broadly and to examine separately every species of inference which seems to involve any extension or any degree of generalization beyond the instantial data comprised within the premisses. We may distinguish four distinctive types of premisses and four corresponding types of conclusion.

The typical premisses are as follows:

I. Every observed P is observed to be Q.
II. A certain proportion of things observed to be P are also observed to be Q.
III. A certain P is observed to be Q.
IV. Observed cases of P_1 are observed to be Q_1 (where P_1 and Q_1 represent approximately specific properties).

The corresponding conclusions are:

I. Every P is Q.
II. A certain proportion of unobserved things that are P are Q.
III. A certain P not observed to be Q is Q.
IV. If there are cases of P_n they are cases of Q_n (where P_n and Q_n represent unobserved properties).

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[1] 'System of Logic,' Bk. III., Ch. III., § 1.

[2] *Ibid.*, Bk. III., Ch. I., § 2 fin.

The inference from premiss I to conclusion I is the typical case of *complete generalization* with which Mill is concerned. Two cases require to be considered: *direct generalization*, and *indirect generalization.*

The inference from premiss II to conclusion II is found in statistical inference. We may call it *statistical generalization.*

The argument from premiss III. to conclusion III. is a simplified statement of the *Argument from analogy*. The distinctive features of the fourth case is that an inference is drawn from an observed concomitance of certain specific properties to some conclusion concerning the connection of other specific properties the occurrence of which need not have been observed. Since ' universals ' rather than particulars are the material of this form of inference, it is suggestive to describe this as the process of ' *universalization* '.

For simplicity we have considered each type of premiss as yielding its own type of conclusion. In actual fact, however, some evidential relation will be found to hold between each type of premiss and each type of conclusion, and the strength of an inductive argument may depend upon a combination of premisses of different types. The examination of these inferences will occupy our attention throughout the remaining chapters of this book. It will be convenient, however, to characterize briefly the four types of inference before proceeding to detailed examination.

1. Complete Generalization.—' *Once a thief always a thief* '. This statement, supported as it commonly is by the assertion ' *That has been invariably the case in my experience* ', is an exemplification of the process of direct generalization. The observer notices that every thief among his acquaintances has permanently resisted every agency of reform. He generalizes so as to embrace within the proposition all other thieves, including presumably those yet unborn. The argument depends, at first sight, upon the number of instances examined. In these instances two properties are invariably conjoined, insensitiveness to the rights of property and insensitiveness to the agencies of reform. It does not depend upon any other specification of these two properties, nor upon any consideration concerning any other properties which the examined cases may possess.

A generalization of this kind may also be made by a less direct route. Instead of basing the conclusion concerning thieves upon data which pertain to thieves alone we might attempt to sustain the argument by reference to other kinds of criminals or by reference to criminals generally. We might go so far as to base it upon the laws of human nature or even by reference to biological laws. But whatever the laws involved they will be complete generalizations reached by some direct induction. Complete induction may, therefore, be subdivided as follows :

Direct Induction, the procedure of proving a generalization of the form *Every P is Q* by reference to data concerning cases of P.

Indirect Induction, the procedure of proving a generalization of the form *Every P is Q* by reference to data concerning either some wider class within which P is contained or certain other classes co-ordinate with P.

It is clear that Demonstrative Induction is a species of Indirect Induction. It is, however, only a special case of a much wider principle and in the sequel it will be important to consider in more detail (*a*) how wide the more comprehensive class must be in order to support a generalization concerning some narrower class, and (*b*) how far knowledge of co-ordinate classes is relevant when we are not in a position to establish a generalization of the more comprehensive type. For example, in order to prove, by the indirect procedure, that thieves are not reformable, is it necessary to know some general law of human nature or is it sufficient to establish some generalization concerning criminals ? And in the absence even of the latter is it relevant, or wholly irrelevant, if we know that some criminals, for instance murderers, cannot be reformed ?

2. Statistical Generalization.—Under this head we may include the obvious extensions of the principles of direct and indirect generalization required to meet cases in which experience does not provide an absolutely constant conjunction of properties. Suppose our observer had known one case of a reformed thief—an isolated exception, let us suppose, to a hundred cases of failure to reform. This is sufficient to destroy his generalization. It would not, however, justify him in trusting his purse to the next thief he meets who claims to be reformed. Though absolute generalization is precluded his experience of the general tendency does not count for nothing. He can at least infer that the chances of reform are extremely small. The more precise formulation of the possible inferences under such conditions belongs to the statistical theory of association and correlation.

8. Analogy.—'Once bitten twice shy'. If this expresses more than an irrational prejudice there must be some logical principle in virtue of which experience derived from one case can be applied more or less directly to another which is similar. From the observation: *I once met a bulldog and it bit me*, and the observation: *here is another bulldog*, we are inclined, if not to assert that this bulldog will bite, at least to avoid giving it the chance. Reasonably or unreasonably we are arguing by analogy. Formally the process may be stated:

A certain thing which is P is Q.
A certain other thing is P.
∴ This other thing is also (probably) Q.

In this case the argument is admittedly weak. It appears to gain additional strength when the known properties common to the two cases are increased. Thus, if we know not only that the two dogs were bulldogs, but also that they were bred from the same stock, that in many details of appearance they were similar, that they lived under similar conditions, were fed in the same way, and so forth, we should feel that the grounds for suspicion were enhanced quite apart from any knowledge of a direct causal connection between any one of these properties and the tendency to bite. Formally our argument is now:

A certain thing which is KLMNOP is Q.
A certain other thing is KLMNOP.
∴ This latter thing is probably Q.

We have increased the degree of analogy between the two cases, and we feel that in so doing we have increased the probability of the conclusion.

It will be noted that the second case may be *anything* which resembles the first. Now, when it is possible to infer from a given case to *any* other case of the same kind it is also possible to infer from the given case to *every* other case of the same general kind.

Analogy would seem, therefore, to provide an alternative route from particular cases to generalization. That is to say, we may attempt to establish a generalization either by increasing the *number of instances* upon which the generalization is based, or by increasing the *number of properties* upon which it may be based. The former may be described as an increase in *extension*, and the latter an increase in *intension*. Schools of thought regarding the logical basis of problematic induction may be broadly divided according as the emphasis is placed upon the extensive or the intensive nature of the evidence. This problem also will occupy our attention in the sequel.

4. Universalization.—One particular form of the procedure of considering the intensive as contrasted with the extensive character of the evidence requires special mention. In the argument from analogy the intension of the instances is increased by reference to properties of other kinds, properties under other determinables than those in which the observer is primarily interested. This is symbolized by the use of separate letters K, L, M, N, O, P, for the properties noted. Another important case of this procedure of increasing the intension is that in which observation relates to a set of instances each of which is characterized by a separate determinate under the same determinable. We observe, for example, that the electrical resistance of a piece of platinum wire varies with its temperature. We notice that at one particular temperature, t_1, its resistance in ohms is r_1; at another temperature, t_2, the resistance becomes r_2; at t_3 the resistance is r_3, etc. After a limited number of observations we may discover the principle of co-variation which enables us to infer without measurement that at t_n the resistance would be r_n. The formula here is

A certain thing which is p_1 is q_1.
" " " p_2 is q_2.
" " " p_3 is q_3.
" " " p_m is q_m.
Therefore if anything is p_n it is q_n.

Universalization, like generalization, varies in degree. In some cases we may be content to infer that some specific value of p will be associated with some specific value of q where these values are not very remote from those directly ascertained. In other cases we may be tempted to the extreme degree of universalization represented by the proposition: *For every different determinate value of P there will be a correspondingly different value of Q.* The conditions of the validity of such inferences belong to the section of inductive theory concerned with the process of establishing 'laws of functional dependence'.

BIBLIOGRAPHICAL NOTES

CHAPTER XII.

In this chapter the main types of inference commonly described as inductive have been classified in the manner suggested by **Johnson** (**Logic,** Part III., Chapter VIII., § 1). After discussion, however, we have found Summary (or ' Perfect ') Induction to be a somewhat trivial operation, and Intuitive Induction to be not strictly a mode of inference. Demonstrative Induction has been assimilated as merely a phase in one form of complete (problematic) induction. The point here is that in direct induction there are two stages, both or only one of which may be problematic. Demonstrative induction is only a case of indirect induction in which one phase is demonstrative. The general lines of exposition in the text are thus in accord with the prevailing opinion that ' Induction ' is, to all intents and purposes, synonymous with ' Problematic Induction '.

The classification of the forms of problematic induction according to the type of premiss from which it starts and according to the type of conclusion to which it proceeds is adopted, with modifications, from **Dr. C. D. Broad's Presidential Address to the Aristotelian Society.** (**Proceedings,** N.S , XXVIII.)

On the Aristotelian doctrine of induction the student should consult **Joseph, Introduction to Logic,** and Aristotle himself. For the historical development of the theory of Induction he should go to the original authors. No satisfactory history of Logic is available : but a good though brief sketch of this history occupies Chapter XXV. of **Stebbing, Modern Introduction to Logic.**

The succeeding chapters of this book being concerned with the special forms of Problematic Induction references will be given in the corresponding bibliographical notes. Direct Induction occupies Chapter XIII., Indirect (including Demonstrative) Induction is dealt with in Chapters XIV. and XV. Universalization (incorporating the ' functional extension ' of demonstrative induction) is discussed in Chapter XVI. and Analogy in Chapter XVII. The subject of Statistical Generalization, requiring a more technical treatment than is possible in this book, is not further developed. For this the student should consult **J. M. Keynes, A Treatise on Probability,** and the address by Dr. Broad referred to above.

CHAPTER XIII.

DIRECT GENERALIZATION.

§ 1. Simple Enumeration as a Psychological Principle.

DIRECT induction has been defined as a process of inference in which, given only instances of things that are both P and Q, we proceed to the conclusion (of greater or less probability) that *Every P is Q*. In the simplest form of the doctrine concerning this species of induction the evidence is presumed to consist in a larger or smaller number of instances in which the two properties occur in conjunction. Nothing is asserted with regard to the other characteristics of the things in question. Moreover, nothing is asserted with regard to the internal constitution of these properties or with regard to the relations between the two properties (except, of course, that neither entails the other).

Historically, the process has been recognized chiefly under the title of *Induction by Simple Enumeration*, a description introduced by Bacon. ' The induction of the ancients ', writes Mill, ' has been well described by Bacon under the name of *Inductio per enumerationem simplicem, ubi non reperitur instantia contradictoria*. It consists in ascribing the character of general truths to all propositions which are true in every instance that we happen to know of. This is the kind of induction which is natural to the mind when unaccustomed to scientific methods. The tendency, which some call an instinct, and which others account for by association, to infer the future from the past, the unknown from the known, is simply a habit of expecting that what has been found true once or several times, and never yet found false, will be found true again '.[1]

It is obvious that in this account of the matter we are presented merely with a psychological description of certain processes in the human mind, a description of conditions under which beliefs may arise. Nothing that has been said gives us reason to suppose that the beliefs which so arise are in any way justified. Even as a psychological account it leaves much to be desired.

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[1] Mill, ' Logic ', Bk. III., Ch. III., § 2.

Absolute clarity would require that the psychological principle here invoked should be distinguished from the principle underlying the association of ideas. The latter merely asserts that *if two things are more or less simultaneously presented to the mind then when one is again presented the second is likely to be recalled.* For example, if a motorist has had an accident at a certain place on the road, a return to this place will tend to remind him of the occurrence. A more specific form of the law is that of the so-called Law of Frequency: *The more frequently two things have been co-presented the greater the tendency for the representation of one of these to revive the thought of the other.* But in neither form does the law directly explain the development of a belief. The motorist on his return remembers the accident. He does not of necessity expect another. There would seem, however, to be a parallel law to the effect that under certain circumstances, *if two things have been frequently perceived together then a subsequent perception of one will engender an expectation of the other.* This principle, which might be described as the *principle of associative belief*, provides a more precise formulation of the doctrine of induction by simple enumeration considered merely as a psychological proposition relating to the causation of beliefs. In this form it is evident that the principle offers no reason whatever for supposing that the beliefs that so arise are rationally justified. It may account for the occurrence of beliefs which by chance are true, but it accounts equally well for the grossest superstition. It cannot contribute anything to a logical theory of induction.

To convert this psychological principle into a justification of the process of generalization we require to find some other principle of a logical character—one in accordance with which it may be held that the data which *cause* a given belief simultaneously afford it a rational justification. According to some authors—for example, Mill—such a principle can only be supplied for certain special cases. This view will be examined in the following chapter. Another possibility is that there is some logical principle which is co-extensive in its application with the principle of associative belief. An example of this type of theory is that which was once sponsored by Mr. Bertrand Russell. This view—though certainly inadequate—is of very distinctive interest. It is perhaps the simplest theory of induction which has ever been proposed. It will serve to elucidate some of the problems of generalization if we examine this theory in some little detail.[1]

§ 2. Simple Enumeration as a Logical Principle.

Mr. Russell's argument is, briefly, as follows: The question is raised whether there is any rational justification for our belief that the sun will rise to-morrow. It is pointed out that there are two *prima facie* grounds for believing this proposition: (*a*) We know that it has risen every day in the past; (*b*) It follows from the laws of motion in their application to the rotation of the earth. The latter reason, however, is a deduction from the laws of motion. But the question whether these laws will continue to hold is of exactly the same kind as the question whether the sun will rise to-morrow.[2]

It is then asserted that the only reason that we have for believing that these laws will continue to hold in the future is that they have held in the past. From this it follows that either there is some principle in accordance with which previous experience justifies generalization or that our belief is baseless.

Mr. Russell then formulates the principle which, if true, would justify the inference. The principle is as follows:

(*a*) The greater the number of cases in which a thing of the sort A has been associated with a thing of the sort B, the more probable it is (if no cases of failure of association are known) that A is always associated with B.

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[1] Russell presents this view in Ch. VI. of the 'Problems of Philosophy'. The logical principle proposed is not strictly co-extensive with the field of the principle of associative belief since it applies only where the things co-presented are *invariably* co-presented. But if the principle is valid for this case it is easily extended to cover statistical arguments, and by this extension it would come to be co-extensive with the psychological law under consideration.

[2] It will be observed that relatively to the conclusion *The sun will rise to-morrow* the first argument is a direct induction, the second is indirect. The two arguments are of the same type only in so far as the principle of direct induction is required in both.

(b) Under the same circumstances a sufficient number of cases of the association of A with B will make it nearly certain that A is always associated with B and will make the general law approach certainty without limit.[1]

This formula is obviously analogous to the laws of association and of associative belief. It differs only in being a law of logic. The law of association asserts that on the basis of co-presentation of items a fresh instance of one item determines the *recall* of the other; the law of associative belief asserts that on the same basis a fresh instance of one item determines an *expectation* of the second; the principle of induction asserts that on the same basis a fresh instance of one item would *rationally justify* the expectation of, or belief in, the presence of the second. In each case the *number* of co-presentations is important. In the case of the law of association the more frequent the co-presentation the stronger is the tendency to recall. In the case of associative belief the more frequent the co-presentation the more confident is the expectation. In the case of the principle of induction, as formulated by Russell, the more frequent the co-presentation the more probable the conclusion. But despite these parallels it is of the utmost importance to distinguish the psychological laws from the logical principle. The former depend upon the nature of the human mind, the latter, if valid, depends upon the constitution of things and upon objective relations between the data and the conclusion.

§ 3. Status and Validity of the Principle.

Deferring consideration of the precise significance of the principle we may inquire: What, if it be true, is its logical status, and on what grounds does it claim acceptance? Certain answers to these questions which are suggested rather than explicitly asserted in Mr. Russell's exposition are of the highest importance. If true, they would establish the theory of induction upon the same impregnable basis as is enjoyed by the fundamental principles of deduction. This is an end aspired to, but never fully attained, in many expositions of the subject.

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[1] In this form the principle is intended to justify an inference from *Every examined A is B* to *Every A is B*. A parallel principle is stated which justifies the inference from *Every examined A is B* to *The next A will be B*. The only difference between the two cases is that the conclusion of the restricted inference has the higher probability. It will, of course, be observed that the 'association' of A and B is not in this case an association of ideas—but a conjunction in fact.

Mr. Russell first points out that the principle is of a kind which cannot be either proved or disproved by experience. It cannot be proved by experience since the principle in question is required in order to warrant any inference from experience. To base the principle itself upon experience would thus involve a vicious circle. It is equally clear that experience cannot disprove the principle. Having, for example, inferred on the basis of invariability in experience the probable conclusion that all men die before they reach the age of one hundred and fifty years, this conclusion would not be invalidated should an exceptional case arise. It would not be invalidated even if some advance in science enabled us to treble the normal span of human life. The original conclusion was that death before a certain age was probable on certain data. It remains probable on these data, whatever probability it may assume on fresh data.[1]

If the principle be thus independent of experience, and is valid, it might naturally be supposed to possess the logical status of other principles which are independent of experience. It might be suggested that it is in the same position as the first principles of deductive logic and of mathematics. That Mr. Russell supposes this to be the case is suggested by the fact that he proceeds immediately to a discussion of these principles—of which, he says, ' the principle of induction is not the only example '.

It is evident, however, that the principle of induction is not in the same position as the principles of deduction. Like the latter, it is not proved or disproved by experience; but, unlike them, it lacks the character of self-evidence. It does not seem to be of the kind of which we may say that it applies not only to the actual state of things, but to all possible worlds alike. That Mr. Russell is conscious of this difference is shown by the fact that he relies upon an entirely different kind of argument in order to commend his principle to our acceptance. His chief argument is that its rejection would lead to scepticism with regard to the generalization of science and of common sense. The validity of this argument must be examined later, but before examining it we must attempt to define with rather more precision the exact force of the principle in question.

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[1] We have seen in Ch. X. that all assertions of probability are relative to the data in this way.

Primarily, the principle tells us that, independently of all other considerations, an increase in the number of instances of a possible generalization yields an increase in the probability of the truth of this generalization. But this in itself does not tell us much. Unless we know both (*a*) the initial probability of the truth of the generalization, and (*b*) the extent to which this probability is increased by a given increase in the number of instances, we are not able to make any determinate assertion with regard to the probability of the conclusion. We cannot even determine whether the generalization is more likely to be true than false. In other words, the principle tells us that the probability of the conclusion is some function of the number of instances examined, but it does not give any determinate information as to the extent to which the probability of the conclusion is increased by additional data. It is consistent with many possibilities which would not yield any sure foundation for the acceptance of scientific laws. Let us suppose, for example, that a single instance of the co-presentation of two properties, A and B, yields a probability of one in four that the generalization *Every A is B* is true (an obvious overestimate); and let us suppose each additional case increases the probability, but only to one-half the amount by which the previous case increased it.[1] The probability of the generalization would then be determined in a manner represented by the series,

$$\tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \ldots$$

where the fractions represent the probability contributed by each examined case. Under such conditions, however many cases we examined, it would for ever remain that the generalization is more likely to be false than true.

The second clause of the principle is perhaps intended to provide against this difficulty. 'A sufficient number of cases will make it nearly certain that A is always associated with B'. In this form, however, the principle is merely tautological. 'A sufficient number of cases' is presumably 'a number of cases sufficient to make the generalization nearly certain'. But what we want to know, of course, is when we have a sufficient number of cases. In the absence of this information we have no adequate basis for generalization.

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[1] Of course this is not a plausible supposition, but in the absence of any guidance from the principle itself we are free to consider any hypothesis.

Upon the precise form of the principle any claim to self-evidence must depend. But it is not by reference to self-evidence that Mr. Russell would defend it. The arguments upon which he chiefly relies do not depend upon its specific form. The main argument is expressed in the following passage:

> 'If the principle is unsound, we have no reason to expect the sun to rise to-morrow, to expect bread to be more nourishing than a stone, or to expect that if we throw ourselves off the roof we shall fall. When we see what looks like our best friend approaching us, we shall have no reason to suppose that his body is not inhabited by the mind of our worst enemy or of some total stranger. All our conduct is based upon associations which have worked in the past, and which we therefore regard as likely to work in the future; and this likelihood is dependent for its validity upon the inductive principle.
>
> 'The general principles of science, such as the belief in the reign of law, and the belief that every event must have a cause, are as completely dependent upon the inductive principle as are the beliefs of daily life. All such general principles are believed because mankind have found innumerable instances of their truth, and no instances of their falsehood. But this affords no evidence for their truth in the future unless the inductive principle is assumed'.

The argument is more intimidating than conclusive. There must be a mistake somewhere since, as we have seen, this principle as it stands does not justify scientific generalization. It is evident, however, that the essential nerve in the reasoning lies in the assumption that no other principle is available.

The issue may be stated in an antilogism:

We cannot hold
(*a*) That scientific laws are known, and
(*b*) That this principle of induction is the only ground for accepting scientific laws, and
(*c*) That this principle is false.

Mr. Russell accepting (*a*) and (*b*) is forced to reject (*c*), i.e. to assert that the principle of induction is true. But for (*b*) he gives no very satisfactory reasons. We can, in common with Mr. Russell, reject many alternative suggestions without feeling confident that no others remain. In principle this is a very difficult position to establish. It would therefore be safer to reason as follows:

Scientific laws are known.
The alleged principle of induction does not afford them a rational foundation.
∴ There must be some other principle which remains to be ascertained.

The search for this alternative will begin in the following chapter.

BIBLIOGRAPHICAL NOTES.

CHAPTER XIII.

The conception of 'Induction by Simple Enumeration' like all vague principles admits of diverse interpretations. In this chapter emphasis has been placed (in accordance with the main tradition) upon that interpretation which assigns most weight to the *number* of instances examined. Some writers (e.g. Mill, in some passages) employ the expression to denote a species of inference in which not even the number of instances is adequately considered. Others again (e.g. **Stebbing, Modern Introduction to Logic,** Chapter XIV, 2) include within the basis of the principle something more than number. How much, and what is included in the definition of the principle depends largely upon whether a psychological or a logical interpretation is placed upon it, and upon whether the author is presenting it as a superstition or as in some measure a valid basis of inference.

Bacon's strictures on the Induction of the ancients are distributed freely throughout his works. The **Advancement of Learning** and **The Novum Organum** contain the most typical and freely quoted passages.

Mill, System of Logic, Book III., Chapter III., where he is echoing Bacon, should be compared with Mill, Book III, Chapter XXI., where he is trying to defend the principle of simple enumeration.

The doctrine expounded by **Russell** is not, it should be noted, explicitly presented as a defence of simple enumeration, though no injustice is done, I trust, in so representing it. The account in **The Problems of Philosophy,** Chapter VI., should be compared with that in **Our Knowledge of the External World,** Chapters I. and VIII.

CHAPTER XIV.

INDIRECT GENERALIZATION: (I) GENERAL PRINCIPLES.

§ 1. Mill on Induction.

'Why is a single instance, in some cases, sufficient for a complete induction, while in others myriads of concurring instances, without a single exception known or presumed, go such a very little way towards establishing an universal proposition? Whoever can answer this question knows more of the philosophy of logic than the wisest of the ancients, and has solved the problem of Induction '.[1]

We have already had occasion to notice the contrast in connection with the principles of demonstrative induction; but at the present stage of the inquiry it must not be assumed that the demonstrative form constitutes the only case to be considered. Mill was writing at a time before the principle of demonstrative induction had received explicit recognition. Its recognition in fact is one of the later developments in the line of thought opened up by his inquiries. It is, moreover, only a special case of the general principle which must rank as one of the most important of Mill's contributions to inductive theory.

Mill as we have already noted defined induction as 'generalization from experience'. More precisely he defined it as 'inferring from some individual instances in which a phenomenon is observed to occur, that it occurs in all instances of a certain class; namely, in all which *resemble* the former, in what are regarded as the material circumstances '.[2]

Certain points in this definition call for notice. By an 'instance' Mill means merely 'a particular'; and by 'an instance in which a phenomenon is observed to occur' we may understand 'a particular observed to possess a certain character'. If these instances resemble others they must, of course, resemble them in respect of some character. Hence Mill's definition, in common with others, refers to the observation of a conjunction of characters. His somewhat roundabout way of stating the matter arises from the fact that Mill has in mind a process of inquiry of the following kind.

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[1] J. S. Mill, 'A System of Logic,' Bk. III., Ch. III., § 3.

[2] *Ibid.*, § 1.

We commence by observing that certain things have a certain property, say P. We then seek for some other property, say A, such that we are enabled to say: Everything that is A will be P. The process of discovering A may be very much more elaborate than the proof of the generalization when A is found. Mill, of course, is right in implying that we do not always start by observing a concomitance, so that all that remains is to prove invariability. Thus, to take one of his own examples, we observe that some things have a crystalline structure, diamonds, lumps of sugar, of quartz and so on. As a result of very prolonged investigation we may discover that all these things have at some time passed in a certain way from a liquid to a solid state. This property is the A which constitutes the basis of resemblance in a material respect, and which enables us to generalize: Every A is P.

Mill next observes that the very definition of induction (assuming that the process is sometimes valid) presupposes an *important assumption with regard to the course of nature* and the order of the universe, viz. that there *are* uniformities in nature, that what happens once under certain circumstances will happen again whenever these circumstances recur; that what is true of any one case is true of all cases of a certain description. It is obvious that this is so, that if a single inductive generalization be valid, then the constitution of nature must be such that there is at least one uniformity. If, moreover, some induction can be made with regard to every sort of occurrence, then every sort of occurrence is an instance of a general law.

At this point Mill's exposition begins to become puzzling. He proceeds to assert both (*a*) that the assumption is the warrant for all inferences from experience, that it is the fundamental principle or general axiom of induction, and (*b*) that it does not offer any explanation of the inductive process. It is on the contrary, he asserts, itself an induction, and by no means the first or the most obvious kind.

It is not possible wholly to exonerate Mill from the charge of confusion of thought on this point, nor wholly to extricate him from the apparently circular argument. On the other hand, it is fairly easy to see that the position he adopted is not completely nonsensical, and to elicit from his statements a doctrine which contains much that is both true and important.

It is true that there is a sense in which the principle that 'Nature is uniform' is both a ground of induction and at the same time offers no explanation of the process of induction. We may, with Johnson, distinguish between the 'constitutive' and the 'epistemic' grounds of induction. A given fact can be said to be the constitutive ground of an inference if it be the case that in the absence of that fact the inference would not be valid. In this sense the fact expressed by the conclusion of any valid argument is part at least of the constitutive ground of the validity of the inference. The epistemic ground for an inference is anything that must be *independently known* before the inference can be carried through. Now, if any inductive inference is valid there must be at least one uniformity in nature. That uniformity would thus be the constitutive ground of that inference. It is obviously not the epistemic ground since if this uniformity were independently known it would not need to be inferred. Similarly, if it be true that an induction can be made with regard to every sort of occurrence then it is the constitutive ground of induction generally that every occurrence is an instance of some law. It may, however, be the case that we can establish many uniformities without assuming this generalization concerning the constitution of nature. It then does not explain induction. Nevertheless, if by any chance there were some way of establishing this proposition about nature independently of a given set of specific generalizations, this proposition would prove very useful in the process of establishing these generalizations. The uniformity of nature would then become part of the epistemic as well as the constitutive ground of these generalizations. There is no doubt that Mill cherished the hope of establishing this proposition.

This, however, is to anticipate. Having defined the problem, Mill proceeds to an examination of the existing generalizations which are in fact admitted by science and to discuss the conditions under which the conviction of their truth may be modified. These generalizations, he notes, fall into two groups; one composed of what are called, in a stricter sense of the term, Laws of Nature; and another composed of generalizations which may be called derivative laws. In the most general sense every uniformity is a law of nature. But we may distinguish uniformities which are not deducible from others from uniformities which are so deducible. The former constitute Laws of Nature in the stricter sense, the latter derivative laws. Thus, if the Newtonian laws of motion be accepted as non-derivative laws, the uniformity expressed by saying that the planets move in elliptical orbits or that water runs downhill are derivative laws. It will be noted that the distinction is relative to our knowledge; it does not reside in the intrinsic nature of the laws themselves. What is a non-derivative law to-day may become a derivative law to-morrow by the discovery of some other law from which it can be deduced.

Now this distinction is connected with a difference between the degrees of assurance we feel in the truth of the laws. Let us suppose that we have collected a certain amount of evidence for three laws, *All A's are B's*, *All P's are Q's* and *All X's are Y's*—the evidence being quite independent for the three cases. Then let us suppose that it is found that the law *Every P is Q* can be directly deduced from the conjunction of the two laws *Every A is B* and *Every X is Y*. How does that affect the situation? It is clear that our assurance of the truth of the assertion *Every P is Q* is increased. It is also clear that, given that there really was evidence for each of these generalizations this increase in assurance is not merely a psychological curiosity. It is rational. Given that the uniformities had a certain probability prior to this discovery the discovery alters this probability. For one thing, the evidence in favour of the assertion that *both* the non-derivative laws are true is now added to the direct evidence for the derivative law. This follows from the nature of deduction, which as demonstrative inference communicates to the conclusion whatever probability there is in the weakest premiss. If there had been no direct evidence for the law *Every P is Q* it would have acquired whatever probability the weaker of the two other laws had possessed. Given direct evidence for the law it now possesses this in addition to the probability communicated by the deductive argument. Not only is this the case; it also communicates in return some measure of its own probability to each of the laws from which it is deduced. Some part of its independent probability is distributed between the two premisses.

Consider another case. As before, there is independent evidence for the three laws *Every A is B*, *Every P is Q*, *Every X is Y*. Now, however, it is found that the two laws *Every A is B* and *Every X is Y* are inconsistent with the law *Every P is Q*. This, of course, does not prove that *Every P is Q* is false. It proves only that the three proposed laws cannot all be true. The evidence in favour of any two of these laws is now pitted against the evidence for the third. It is an antilogism of probabilities. Special interest attaches to the case in which a narrow generalization is inconsistent with a wider generalization. The law that matter gravitates has acquired a high probability from the study of the behaviour of water in hilly country, of fruit dropping from trees, of crockery falling from tables and so on. The behaviour of balloons and aeroplanes provides a *prima facie* difficulty. In the absence of adverse considerations the evidence from things other than balloons and aeroplanes goes to prove that balloons and aeroplanes gravitate in spite of appearances. The evidence from balloons and aeroplanes at least suggests that matter does not gravitate. But the wider generalization weakens the narrower to a greater extent than the narrower weakens the wider. The breadth of the generalization concerning other forms of matter has an almost overwhelmingly destructive influence upon the probability of the conclusion to be drawn from the observation of balloons; the narrowness of the latter generalization makes its influence upon the wider law practically negligible.

So far as the narrow generalization '*balloons do not gravitate*' does weaken the generalization '*matter gravitates*' its influence is not restricted to this generalization alone. It weakens every generalization comprised within it. If balloons do not gravitate that is *some* evidence that apples do not either. This can be seen in the case in which the conflict occurs between generalizations of more equal independent force. The discovery that not all swans are white not only weakens any proposed generalization that all species of animals are constant in colouring; it lowers the probability that all crows are black. It weakens the force of every narrower generalization concerning constancy of colour.

Granting that in some manner every instance of a generalization increases its probability, we may say that the final probability attained does not depend upon these instances alone. Every instance of a general law not only strengthens its own specific law, it strengthens every law of a parallel kind. More especially, the wider the generalization the greater its confirmatory value to any narrower generalizations which fall within its scope.

There are thus two ways in which a generalization may be reached—direct and indirect. By the direct method, each specific law depends upon instances of the specific law in question. By the indirect method its probability is enhanced by establishing a law of a parallel kind or of a generalization from which it may be deduced. In Mill's words: 'The stronger inductions are the touchstone to which we always endeavour to bring the weaker. If we find any means of deducing one of the less strong inductions from stronger ones, it acquires, at once, all the strength of those from which it is deduced; and even adds to that strength; since the independent experience upon which the weaker deduction previously rested becomes additional evidence of the truth of the better established law in which it is now found to be included'.[1]

'It may be affirmed as a general principle, that all inductions, whether strong or weak, which can be connected by ratiocination, are confirmatory of one another; while any which lead deductively to consequences that are incompatible become mutually each other's test, showing that one or other must be given up, or at least more guardedly expressed. In the case of inductions which confirm each other the one that becomes a conclusion from ratiocination rises to at least the level of certainty of the weakest of those from which it is deduced; while in general all are more or less increased in certainty'.[2]

In drawing attention to these reciprocal influences of originally independent generalizations Mill laid the foundations of a theory of indirect generalization. But at this point he succumbed to an unfortunate temptation. It occurred to him that 'If a survey of the uniformities which have been ascertained to exist in nature should point out some which, as far as any human purpose requires certainty, may be considered quite certain and quite universal, then by means of these uniformities we may be able to raise multitudes of other inductions to the same point in the scale. For if we can show, with respect to any inductive inference, that either it must be true, or one of these certain and universal inductions must admit of an exception, the former generalization will attain the same certainty, and indefeasibleness within the bounds assigned to it, which are the attributes of the latter. It will be proved to be a law; and if not a result of other and simpler laws, it will be a law of nature.

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[1] J. S. Mill, 'A System of Logic', Bk. III., Ch. IV., § 3.

[2] *Ibid.*

'There are such certain and universal inductions; and it is because there are such, that a Logic of Induction is possible'.[1]

In point of fact, however, if there be any such law it abolishes the logic of induction altogether. Once the law is ascertained scientific method becomes a purely deductive operation. We may, however, with advantage follow Mill's argument a little further.

§ 2. The Law of Causation.

The 'phenomena' of nature occur in two relations, simultaneity and succession. Moreover, according to Mill, every phenomenon is related in a uniform way to some simultaneous phenomenon or to some which it precedes or follows. The general uniformity of Nature is thus constituted by a tissue of uniformities of co-existence and uniformities of sequence. There is, however,—we are still quoting Mill—an important difference between the two cases. The uniformities of co-existence relate only to the spatial and numerical properties of things. The uniformities of sequence pertain to every property. Thus, if we know of anything whatever that it is a perfectly spherical solid then we know that it will be exactly two-thirds the volume of a cylinder of equal height and diameter. These two properties invariably 'co-exist'.[2] On the other hand, given that the solid is red or hot or has any other non-spatial or non-numerical property there is no reason to suppose there is any other property with which such properties invariably co-exist. The most that we can say, therefore, with regard to uniformities of co-existence is that everything has *some* properties which invariably co-exist with others. In the case of uniformities of sequence, however, there is an important general truth which applies to every property of everything that exists. This is embodied in the law of causation. Everything has a cause. This is a generalization about sequences. It implies not only that some sequences are invariable but also that the occurrence of every property is connected by invariable sequence with the occurrence of some other property.[1] It is this law which is selected by Mill to be the touchstone to which all independent generalizations must be brought. It provides the guarantee of every other generalization.

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[1] J. S. Mill, 'A System of Logic', Bk. III., Ch. IV., § 3.

[2] Mill holds that these laws of co-existence and the 'axioms' from which they are deduced are established by pure induction. But this special doctrine is not relevant to our present subject.

In assigning such a key position in the theory of induction to the law of universal causation it might seem that Mill had involved Logic in an unfortunate entanglement with metaphysical theory. But it is possible to isolate the area of conflict and to state Mill's position in a manner that frees his theory of induction from any distinctive metaphysical basis. 'Causal connection', says Mill, 'consists in invariability of sequence'. This admittedly is a controversial statement. But let us suppose that he is right. The doctrine eliminates causality from the ultimate categories of thought The word 'causation' ceases to signify anything distinctive. What it expresses can be stated in other familiar terms, in short, in terms of 'uniformity' and 'sequence'. In this there is nothing specifically metaphysical. But let us now suppose that he is wrong and that we could prove it to his satisfaction. To this Mill might well reply that, even so, his position was unaffected. If 'causation' means anything more than invariability of sequence whatever more it means is irrelevant to Logic. Given invariability of sequence he offers to provide us with a basis for induction.

It is therefore possible to state Mill's position without discussing in detail the nature of causation. We have to examine the possibilities of induction given the general proposition that phenomena occur in invariable relations. It will be convenient, however, to state Mill's position in the words that he himself employs; using, that is to say, the word 'causation' for invariability of sequence.[2]

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[1] The expression 'occurrence of a property' is employed to replace Mill's 'phenomenon'. By the 'occurrence of a property' is meant its manifestation in a place or at a time or as belonging to something or other.

[2] Of course, had Mill been willing to distinguish causality from invariability of sequence it is doubtful whether he would in fact have

'Everything has a cause'. We must now define with more precision the principle upon which Mill relies. Sometimes he speaks of 'a cause' loosely, meaning simply 'an invariable antecedent'. On this interpretation the causal principle would be as follows:

> *With regard to every property by reference to which a phenomenon may be defined there is some other property the occurrence of which invariably precedes it.*

I.e. for any property, *p*, there is an antecedent, *a*. Or, given that *p* occurs we may infer that *a* has occurred. Since there are passages in Mill's exposition in which this principle appears to be assumed it will be convenient to refer to it as the *principle of invariable antecedence.*

More frequently, however, Mill is less unguarded in his statements. He is in fact troubled throughout by the problem of 'plurality of causes'. The same phenomenon may have alternative causes. If death is sometimes due to poisoning and sometimes due to gunshot can we say that it has any invariable antecedent? Possibly we can, but to avoid prejudging the issue we require a different interpretation of universal causation. A 'cause' of a phenomenon may be defined as an antecedent upon which the phenomenon in question is invariably consequent. The principle then becomes:

> *With regard to every property by reference to which a phenomenon may be defined there is at least one other property upon the occurrence of which the phenomenon in question is invariably consequent.*

I.e. for every property, *p*, there is at least one other property, *a*, upon which *p* invariably follows. Here we note that, although *a* cannot be inferred from *p*, *p* can always be inferred from *a*. This we may call the *principle of invariable consequence.*

A 'cause' in Mill's, as in any other, sense must be distinguished from a 'condition'. Taking the cause to be that upon which the phenomenon is invariably consequent, the antecedent need not be, and perhaps never is, absolutely simple. It may

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formulated his distinctive theory A good deal of the plausibility in the assumption that everything happens in accordance with some law is derived from the supposition that this is in fact equivalent to the assertion that everything has a cause.

involve a collocation of properties such that in the absence of any one of these the consequent would not occur. We thus obtain the distinction between:

The complete cause, i.e. the sufficient condition of *p*.

The cause factors, i.e. the various necessary conditions of *p* (of which the sufficient condition is composed).

We so obtain as a final statement of the principle of invariable consequence:

> '*For every phenomenon p there is a set of necessary and sufficient conditions upon which p is invariably consequent*'.

This is the supreme principle, according to Mill, upon which the discovery and proof of specific uniformities depends. The process from this point becomes wholly deductive. We may consider, for example, the manner in which we might ascertain that the necessary and sufficient conditions for the life of a flower consist of seed, soil, water, air and sunshine. We know to begin with, let us suppose, that the probable conditions lie somewhere in a group of circumstances containing a number of these elements and others such as thunder and lightning and moonshine. By removing one of these items at a time we may progressively sort them into two groups, relevant and irrelevant. The plant flourishes whether exposed to the moon or not. The moon therefore is not a necessary condition. It languishes in the absence of the sun, so the sun is a necessary condition. But sunshine without water is too much of a good thing, so sunshine is not a sufficient condition. So with each factor in turn, until we have segregated a compact group of conditions which, being severally necessary and jointly sufficient, constitute the cause. Hence the law is established: Whenever there is a seed, in soil appropriately watered and provided with sunshine, there will be a plant.

Such considerations suggest that it should be possible to formulate a set of definite rules of procedure which, methodically adhered to, would progressively reveal and conclusively prove all the laws of nature. The problems of science would then become amenable to a procedure as simple and as elegant as the application of the rules of syllogistic inference. This possibility had occurred to Bacon. It was developed by Mill. More precise formulations were later provided by Johnson. The examination of such 'inductive methods' will occupy us in the following chapter. There remains to be considered, however, an outstanding question of principle.

§ 8. The Evidence for Universal Causation.

It has been seen that one of the central features of Mill's account of induction, ignoring minor refinements, qualifications and possible inconsistencies, is the doctrine that there is a law to the effect that the occurrence of every property is connected in a uniformity of sequence with the occurrence of some other property. Given this as a major premiss it is possible by methods of elimination to determine in a demonstrative way innumerable special generalizations.

Clearly, whatever reliance may be placed upon the generalizations so attained will depend upon the truth of the major premiss so employed. How is this premiss established?

It is established in the last resort, says Mill, by the procedure of simple enumeration. If we are permitted to employ a direct inductive method in this unique case we can dispense with it elsewhere. We are justified in doing so because the objections which can be urged against it in other cases fail to arise in this special connection. Not only is the method free from positive objections in this particular case, but it provides here a very much closer approximation to certainty than would be possible were the method applied directly to the proof of specific uniformities. In virtue of the indirect procedure this approximation to certainty can nevertheless be transferred to the narrower generalizations with which science is concerned.

Mill's arguments for this position fall into two groups. Principally they are arguments to show that the obvious objections to the procedure of simple enumeration do not apply in this particular case. A second, somewhat vaguer, set of arguments is offered in defence of the force and validity of this direct inductive procedure.

The most fundamental objection to the process of simple enumeration is that it sometimes leads to admittedly false conclusions. 'A mode of concluding from experience must be pronounced untrustworthy when subsequent experience refuses to confirm it. According to this criterion, induction by simple enumeration—in other words, generalization of an observed fact from the mere absence of any known instance to the contrary—affords in general a precarious and unsafe ground of assurance;

for such generalizations are incessantly discovered, on further experience, to be false '.[1]

Now this at least has never been shown to be the case with regard to any assertion of the form '*X has a cause*'. We can, in fact, go further. An instance which invalidates some specific law often provides additional evidence for the general law of causation. This is the case in all generalizations which fail on account of 'counteracting causes'. The law of causation, at any rate, cannot be interfered with by a counteracting cause.

This argument, of course, does not amount to much. That no proposition of the form '*X has a cause*' has ever been disproved is to be expected on the ground of its very indeterminate nature. It means, on Mill's view, that there is *some* property upon the occurrence of which X is invariably consequent. To disprove this it would be necessary to show, with regard to every property whatever, that its occurrence has sometimes not been followed by X. Nothing has been so exhaustively studied as to justify any such assertion.

Mill relies, however, chiefly upon another consideration in exempting simple enumeration from the charge of unreliability to which it is open in more specialized cases. This he expresses by saying that 'the precariousness of the method of simple enumeration is in inverse ratio to the largeness of the generalization'. What Mill means by this is by no means clear. By the 'largeness of the generalization' he cannot mean the extensiveness of the class concerning which the generalization is made. It is obviously false to say (without further explanation) that the generalization *All material bodies gravitate*, is less precarious than the narrower generalization *All stones gravitate*. It is not improbable that three distinct considerations were in his mind. The first was the observation that there are certain highly trustworthy generalizations concerning properties that are very general in the sense that they characterize a very wide range of particulars—properties such as spatio-temporal and numerical attributes. Mill, as we have noted, refused to recognize the distinctive nature of those generalizations and supposed them to be reached by problematic induction—i.e. by simple enumeration. The second consideration was that which we have noted in saying that relatively indeterminate generalizations are less precarious than specific generalizations. It is obviously safer to assert that every material thing is coloured than to say that every material thing is red. A third possible consideration relates to the greater probability which attaches to generalizations when there is 'largeness' not in what is inferred but in the evidence from which it is inferred. Thus, although (other things being equal) it is more precarious to infer that *All material bodies gravitate* than merely that *All stones gravitate*, it is plausible to suppose that the former generalization would be safer on the basis of a given number of instances selected from a great variety of materials than the narrower generalization would be on the same number of instances selected from stones alone.

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[1] J. S. Mill, 'A System of Logic', Bk. III., Ch. XXI., § 2.

Nevertheless, whatever may have been in Mill's mind it is difficult to find in his statement any positive reason for attributing a reasonably high probability to the conclusion he wishes to draw. The reference to generalizations concerning spatio-temporal and numerical properties contributes nothing to the case owing to the distinctive logical status of these generalizations. The indeterminateness of the law of causation explains the difficulty of disproving it without contributing anything positively in its favour. The considerations with regard to the generality of the evidence may later prove to be of importance, but they are much too vaguely presented in Mill's account of the matter to provide a satisfactory justification of the process of simple enumeration. Moreover, so far as they are relevant to the problem of induction they will be found to involve a radical departure from the principle of enumeration. They relate to the influence of *variety* in the instantial premiss—whereas in the doctrine of simple enumeration the emphasis is placed simply on number alone.

Two approximations to a positive argument for the validity of the process of simple enumeration are offered by Mill. The first is the appeal to established truth. Granting that the method is precarious Mill proceeds, 'Still, however, it affords some assurance, sufficient, in many cases, for the ordinary guidance of conduct. It would be absurd to say that the generalizations arrived at by mankind in the outset of their experience, such as these, Food nourishes, Fire burns, Water drowns, were unworthy of reliance'.[1] This, however, leaves much to be desired. It remains to be shown that these generalizations are in fact attained by simple enumeration alone. Even admitting that they were, it is a far cry from these to the law of universal causation.

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[1] J. S. Mill, 'A System of Logic', Bk. III., Ch. XXI., § 2. This, it will be noticed, is the argument used by Russell in defence of his principle of induction. It is open to similar objections here.

Mill's second argument involves the introduction of a validity clause into the statement of the principle. In its cruder forms induction by simple enumeration requires only (1) that the proposed generalization *Every A is B* be exemplified in known instances, i.e. that there are examined A's that are B's.

(2) That there should be no known instances of A that are not B.

To this Mill adds a third condition which confers validity:

(3) That if there are any A's that are not B we are likely to know of them.

There is a danger of interpreting this condition so as to reduce the principle of induction to a mere tautology. Clearly, this proviso, combined with the fact that we know of no exceptions, yields, as an immediate inference, the conclusion that there probably are no exceptions to this law. This in turn is equivalent to the statement that Probably every A is B. But if we include this among the premisses of induction, what remains to be added in the conclusion? This is precisely what we want to prove. It is difficult to interpret the proviso in any way which both escapes this objection and gives any added force to the ordinary principle of enumeration. We must, therefore, conclude that whatever Mill may have done to free the principle from specific objections he has done little to invest it with positive inferential force.

§ 4. An Evaluation of Mill's General Theory.

It is easy to criticise Mill. But when criticism is finished it remains to be admitted that Mill left inductive theory in a better position than that in which he found it. His exposition has proved a convenient starting-point for nearly every subsequent discussion; and almost every important advance may be not unfairly represented as a direct development of a suggestion to be found somewhere in his system.

We may perhaps select as the most significant of Mill's contributions to inductive theory the emphasis placed upon the distinction between direct and indirect methods of supporting a generalization. The generalization *Every A is P* may be supported merely by finding instances of A's that are P. This procedure is direct. It may also be supported indirectly, by establishing parallel laws, such as *Every B is P*, *Every C is P*, etc., where A, B, C, etc., constitute, in virtue of other similarities, species of a wider class say, M.

There are, however, two ways in which the generalization, *Every B is P*, *Every C is P*, etc., may effect the probability of *Every A is P*. The first lies in the fact that, independently of evidence concerning A, these parallel generalizations afford evidence for the conclusion *Every M is P*. From this *Every A is P* would follow by strict deduction. That is to say, whatever probability attaches to the conclusion, *Every M is P*, in virtue of the data concerning B, C, D, etc., is added to whatever probability *Every A is P* may have acquired directly from the observation that every examined A is P. There is a second way in which these parallel generalizations influence the probability that *Every A is P*. The observation that *Every examined B is P*, *Every examined C is P*, etc., severally and collectively increase the probability of *Every A is P* independently of any generalization to the effect that *Every M is P*. This is most clearly seen in the case in which it is known that this wider generalization is false. Thus in discussing family traits we might argue as follows: *The Smiths are a clever family. Old John Smith's children were all clever; Henry Smith's were: so were George's, William's and Robert's. It is true that Stephen Smith's children were not at all clever; but taking the Smiths as a whole it is likely that James Smith's children will be clever.* Admittedly the argument would have had greater force had Stephen's children also conformed to the rule, but the inference has more force than it would have had in the absence of information concerning the general tendency in the family.

Both procedures are indirect, and both may be described as processes in which a narrower generalization is tested by the touchstone of a wider. It is unfortunate that Mill attended almost exclusively to the first case alone, and made it the basis of his final theory. It was more unfortunate that he failed to realize that whatever procedure we adopt the conclusion will at best be problematic. He tried to extract conclusions that are certain from data which yield only probabilities.

Another unfortunate step was to seek for one comprehensive generalization by reference to which every other generalization might be tested. He failed to observe that the connection between precariousness and breadth of generalization was not the simple one of inverse variation. It would be truer to say, as a first approximation, that the highest probability attaches to certain generalizations which occupy an intermediate position in the scale. Generalizations of a highly specific kind (so far as they depend upon their own specific evidence), and universal propositions about things in general, are both more precarious than generalizations concerning well-defined and comprehensive groups. Thus we find among the most trustworthy generalizations of science those relating to the general properties of solids, of gases, of organic matter, and so forth. These are more certain than any generalizations concerning sulphur or hydrogen, so far as the latter depend solely upon observations relating to sulphur or to hydrogen. They are also more probable than anything so very general as the Law of Universal Causation.

It must be added that Mill failed to discover any logical principle which confers logical validity upon the principle of simple enumeration. It can scarcely be doubted, however, that he was right in contending that some such principle there must be, and that in the absence of any alternative we are justified in employing it.

If in consequence of the deficiencies in his doctrine he failed to establish the Law of Universal Causation it does not follow that the existence of any such law can be positively denied. In the absence of arguments for indeterminism the considerations upon which Mill relies at least confer upon the law of causation a measure of probability. This measure may well be greater than pertains to many more specific laws. It follows that by the indirect procedure the probability of the latter can at least be raised. They attain at least, that is to say, to the probability of determinism.

Mill over-estimated this probability, but it remains of interest to examine the procedure by which this additional strength can be conferred. If the methods be technically sound they may then be more effectively employed by the substitution of a more restricted major premiss.

BIBLIOGRAPHICAL NOTES.

CHAPTER XIV.

The sketch of Mill's theory in this and the following chapter should not be accepted, even by the elementary student, as a substitute for reading Mill himself. There are many interpretations of Mill's doctrine and any selection from his statements is bound to bias the interpretation adopted. The selection here made is intended merely to illustrate the conception of indirect induction : it is not so much concerned with Mill for his own sake. The whole of Book III. of 'The System of Logic' should be read, though the chapters most relevant to the topics here discussed are Chapters III., IV., V., XXI.

Other interpretations and criticisms of Mill should be studied. The exposition of the theory of Induction since Mill divides into two main streams. The one is generally sympathetic and reflects his influence, though becoming increasingly critical and introducing more and more fundamental lines of reconstruction. This stream proceeds from Bain, Mill's contemporary, who merely dotted the i's and crossed the t's of Mill, through Venn to Johnson. The other stream is more critical but less constructive. The remarks of Mr. J. M. Keynes concerning contemporary logicians apply to most writers since Mill : 'They content themselves for the most part with the easy task of criticizing Mill or the more difficult one of following him', **Treatise on Probability,** Chapter XXIII., **'Some Historical Notes on Induction'.** This all-too-brief chapter should be read by every student.

CHAPTER XV.

INDIRECT GENERALIZATION: (II) DEMONSTRATIVE METHODS.

§ 1. General Presuppositions of Mill's Experimental Methods.

It has been seen, in the preceding chapter, that direct induction commonly yields only a relatively low degree of probability, more especially when employed to establish a highly specific generalization in isolation from all parallel laws; but that the probability so attained may be considerably increased by showing that the law in question is deducible from more general laws which have also been reached by direct induction from independent data. This fact, we saw further, suggested to Mill the possibility that there might be at least one very general law through which all other laws could be supported. He tried to show that the Law of Universal Causation would serve the purpose, and that in establishing the Law of Causation the principle of simple enumeration (or of direct induction) provided an adequate logical basis.

We have now to examine the procedure by which this, or any similar, law can be employed for the purposes of an investigation into special cases, i.e. to examine in detail the methods of indirect induction. In order to isolate the problems peculiar to this procedure we may ignore in this chapter all the doubts that may attach to the principle of simple enumeration and to the Law of Universal Causation. Even though it be that no adequate grounds have been given for the acceptance of this principle it may very well be the case that we are in fact justified in using it. Its validity follows, perhaps, from some quite different principle which remains to be discovered. In the same way, although Mill failed to prove the Law of Universal Causation this law may quite well be true. It remains, perhaps, merely to think of the appropriate principles and to marshal the evidence in a different way.

For the purposes of the present chapter, therefore, it will be assumed that the Law of Universal Causation has at least a higher measure of probability than many specific generalizations. Granted this, Mill's aim is to show how whatever assurance may be felt in this law may be communicated to the specific generalizations which exemplify it, and to devise certain special formulæ by which the general law may be applied to the investigation of particular cases. This is the purpose of his ' Four Experimental Methods ', or ' Methods of Direct Induction '. Despite his terminology it will be noticed that the use of these methods is essentially *indirect*, and commonly not experimental. Applying in detail what is already asserted in general in the Law of Causation they are essentially methods of demonstration. They are in intention as demonstrative as the figures of the syllogism with which in fact Mill compares them. One of the clearest statements of their purpose is that which Mill gives in the following passage :

> ' The business of Inductive Logic is to provide rules and models, (such as the Syllogism and its rules are for ratiocination) to which if inductive arguments conform, those arguments are conclusive, and not otherwise. This is what the four methods profess to be, and what I believe they are universally considered to be by experimental philosophers who had practised all of them long before any one sought to reduce the practice to theory '.[1]

The fact that their premisses may be highly problematic in no way detracts from their demonstrative character. They are demonstrative just in so far as the premisses communicate without loss to the conclusion whatever degree of probability they possess. This we have seen to be the distinguishing feature of demonstrative inference.[2]

Before examining the validity of these Methods we may with advantage note with some care Mill's own exposition. He formulates his problem in the following way :

> ' That every fact which begins to exist has a cause, and that this cause must be found in some fact or concourse of facts which immediately preceded the occurrence, may be taken for certain. The whole of the present facts are the infallible

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[1] Mill, Bk. III., Ch. IX., § 6.

[2] *Supra*, Ch. II.

> result of all past facts, and more immediately of all the facts which existed at the moment previous. Here, then, is a great sequence, which we know to be uniform. If the whole prior state of the entire universe could again recur, it would again be followed by the present state. The question is, how to resolve this complex uniformity into the simpler uniformities which compose it, and to assign to each portion of the vast antecedent the portion of the consequent which is attendant on it '.[1]

The process of resolving this complex uniformity he finds to consist in two operations of analysis, one purely mental, the other dependent upon comparative observation and experiment. This distinction is important since it contains some of the implicit assumptions not obvious in the condensed formulation of the methods.

This operation (of resolution) is more than a merely mental analysis.

> ' No mere contemplation of the phenomena, and partition of them by the intellect alone, will of itself accomplish the end we have now in view. Nevertheless, such a mental partition is an indispensable first step. The order of nature, as perceived at a first glance, presents at every instance a chaos followed by another chaos. We must decompose each chaos into single facts. We must learn to see in the chaotic antecedent a multitude of distinct antecedents, in the chaotic consequent a multitude of distinct consequents. This, supposing it done, will not of itself tell us on which of the antecedents each consequent is invariably attendant. To determine that point, we must endeavour to effect a separation of the facts from one another, not in our minds only, but in nature. The mental analysis, however, must take place first '.[2]

> ' The different antecedents and consequents being, then, supposed to be, so far as the case requires, ascertained and discriminated from one another, we are to inquire which is connected with which. In every instance which comes under our observation, there are many antecedents and many consequents. If those antecedents could not be

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[1] Mill, Bk. III., Ch. VII., § 1.

[2] *Ibid.*

> severed from one another except in thought, or if those consequents never were found apart, it would be impossible for us to distinguish (*a posteriori* at least) the real laws, or to assign to any cause its effect, or to any effect its cause. To do so, we must be able to meet with some of the antecedents apart from the rest, and observe what follows from them; or some of the consequents, and observe by what they are preceded. We must, in short, follow the Baconian rule of *varying the circumstances.* This is, indeed, only the first rule of physical inquiry, and not, as some have thought, the sole rule; but it is the foundation of all the rest '.[1]

This is the general conception of the course of Nature presupposed by the ' Experimental ' Methods.

§ 2. The Four Experimental Methods.

Mill's Methods are four in number, The Method of Agreement, The Method of Difference, The Method of Residues, and the Method of Concomitant Variations. Since, however, the Method of Agreement assumes two forms, according as it is applied singly to positive cases only, or in a double way to positive and negative cases jointly, Mill formulates five ' Canons ' corresponding to the Methods, in the following way: [2]

Method of Agreement, Single Application —First Canon.
Method of Agreement, Double Application—Third Canon.
Method of Difference —Second Canon.
Method of Residues —Fourth Canon.
Method of Concomitant Variations —Fifth Canon.

Of the five Canons Mill seems to have regarded that of the *Method of Agreement* and that of the *Method of Difference* as the most fundamental. It is presumed in each that we have performed the requisite mental analysis of the situation to which the methods are applied and that we are in a position to select instances of the phenomenon under investigation with full knowledge of the relevant circumstances. We then proceed to make a selection of such instances and of certain suitable cases in which the phenomenon is absent.

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[1] Mill, Bk. III., Ch. VII., § 2.

[2] Mill's statement makes it by no means clear how the Canons are related to the Methods, with the result that logicians differ with regard to which of the Canons fails to correspond to a distinct ' method '. (Cf. Stebbing, ' Modern Introduction to Logic ', Ch. XVII.) A detailed study of Mill's statements suggests that whichever Canon *ought* to be excluded, he himself, *intended* to exclude the third as being only a special case of the first.

The Method of Agreement.—To begin with, we may restrict ourselves to a selection of instances of the presence of the phenomenon. Then, if the circumstances of these instances conform to a certain requirement, we are able to effect a generalization. The requirement is formulated in

The First Canon. (Method of Agreement.)

If two or more instances of the phenomenon under investigation have only one circumstance in common, the circumstance in which alone all the instances agree is the cause (or effect) of the given phenomenon.

The principle is roughly illustrated by the way in which an examiner, noting the frequency of an unusual example employed in a number of examination scripts, may be led to suppose that its source is in a certain text-book. He might come to this conclusion if he found that the students came from various schools, differed in ability and in all other circumstances, and that the only common feature in the situation was that a certain text-book was in use in every school in which the students using the example were known to have been instructed.

An example from science selected by Mill is the case in which we are led to suppose that whenever a fatty oil or a fat is mixed with an alkaline substance a soap is produced. All sorts of fatty oils are mixed with all sorts of alkalis, the only common consequent is the production of a ' greasy and detersive or saponaceous substance '. Formally the inference may be symbolized as follows:

ABC . . . *abc.*
ADE . . . *ade.*
∴ A . . . *a.*

Here the capital letters represent antecedents, and the small letters consequents. No significance is to be attached to the fact that the same letters or the same number of letters appear in both groups. In reasoning from effects to causes it is presumed that we start from cases of *a* in various contexts *bc*, *de*, etc., and discover the corresponding antecedent A. In the reverse case we start from examples of the antecedents and explore the consequents.

Considered as a process of *proof* the argument, according to Mill, is as follows : The method is one of *elimination*. Whatever is absent in a given case obviously cannot be present in every case. Whatever circumstance can be eliminated cannot be connected with the phenomenon by any law. Hence *b* and *c* cannot be the invariable effect of A, because in one case these are absent when A is present. Similarly *d* and *e* can be eliminated because they are absent in the other case. A is present when they are absent. If, therefore, A has any invariable consequent it must be *a*. The validity of this argument will be examined later.

The Method of Difference.—More important are the inquiries in which we compare cases in which the phenomenon is present with cases in which it is absent. We select some situation in which a phenomenon occurs, and another situation in which it does not occur. If, then, we find that the two situations differ in their antecedents only in one particular feature, that feature must be a condition of the phenomenon in question. The principle is formulated in

The Second Canon. (Method of Difference.)

If an instance in which the phenomenon under investigation occurs, and an instance in which it does not occur, have every circumstance in common save one, that one occurring only in the former ; the circumstance in which alone the two instances differ is the effect, or the cause, or an indispensable part of the cause of the phenomenon.

As Mill remarks, it is almost superfluous to give examples of a logical process to which we owe so many of the conclusions we draw in daily life. We observe, for example, that before consuming a certain article of food a man is in good health. Immediately after taking it he is ill. If we are satisfied that this is the only circumstance which has changed, we infer, by the method of difference, that the food was the agent responsible for the state of ill-health.

In Mill's symbolism the Method of Difference assumes the following form:

ABC . . . *abc*.
BC . . . *bc*.
∴ A . . . . . *a*.

The method has as its foundation the principle that whatever antecedent cannot be eliminated consistently with the retention of the phenomenon is connected with the latter by some law.

These two canons being taken as fundamental, Mill formulates three others which occupy a subsidiary place or are applicable in special circumstances.

The Joint Method of Agreement and Difference.—This method is not, as its name suggests, a combination of the two foregoing methods. It is better described by Mill as a double employment of the Method of Agreement. One set of instances is selected in which the phenomenon is present, and one set in which it is absent. Then if, as in the simple Method of Agreement, the former agree only in the presence of one antecedent, and if the latter agree only in the absence of that antecedent, then the antecedent in question is causally connected with the phenomenon. This is expressed by Mill in

The Third Canon. (Joint Method.)

If two or more instances in which the phenomenon occurs have only one circumstance in common, while two or more instances in which it does not occur have nothing in common save the absence of that circumstance, the circumstance in which alone the two sets of instances differ is the effect, or the cause, or an indispensable part of the cause, of the phenomenon.

Mill proposes no method for symbolizing this method. It might perhaps be represented as follows, where Ā represents the absence of A and *ā* the absence of *a*.

ABC . . . *abc*.   ĀFG . . . *āfg*.
ADE . . . *ade*   ĀHK . . . *āhk*.

It should be noticed that in this method no pair of instances, one positive and the other negative, conforms to the conditions of the pure method of difference. Otherwise the Joint Method would serve no special purpose. Comparing this with the example given of the pure method of Agreement, it is evident that the examiner's theory as to the origin of a given example would be strengthened if the only relevant feature in common to all students who did *not* give the example in question was that they came from schools in which the given text-book was *not* employed. There are, however, certain difficulties with regard to this method which will call for notice later.

The Method of Residues.—This method, on Mill's view, is a modification of the Method of Difference. It differs from the latter only in the fact that the negative instance is obtained, not by observation, but by inference. We know by observation and experiment that a complex phenomenon, *abc*, is the effect of a complex antecedent, ABC. The application of one of the fundamental methods shows that B is the cause of *b*, and that C is the cause of *c*. We can thus infer that BC together and in isolation would produce *bc*. We thus obtain the following premisses:

By observation and experiment	ABC . . .	*abc*.
By deduction	BC . . .	*bc*.

These instances conforming to the conditions of the Method of Difference yield the conclusion A . . . *a*. The principle of this method is formulated in the

The Fourth Canon. (Method of Residues.)

Subduct from any phenomenon such part as is known by previous inductions to be the effect of certain antecedents, and the residue of the phenomenon is the effect of the remaining antecedents.

Of all the methods of investigating laws of nature this is regarded by Mill as the most fertile in unexpected results and the most valuable for revealing inconspicuous causal factors. The classical examples are provided by astronomy and chemistry. These sciences contain many examples of causal agents being isolated as residual factors in situations otherwise fully explained by reference to known laws. In these cases, however, it is rarely that the several effects and the several 'circumstances' stand out as groups of isolated and distinguishable items, in the manner suggested by Mill's symbolism, in which each effect stands paired with a corresponding cause. A rough approximation to the requirements is provided by the case in which, ignoring all quantitative considerations, we may analyse a movement in three dimensions of space, and assign a distinct cause to the movement in each direction. Thus, a shell is projected in a northerly direction parallel to the surface of the earth. Its course is found to have a northerly trend *c* which can be referred to the construction and position of the instrument of projection, *C*. It has a westerly trend *b* referrible to the influence of an east wind, *B*. But we note a downward trend *a*. This demands the presence of a third causal factor *A*, the gravitational influence of the earth. Mill's method is intended to apply to such cases where it is impossible to find examples satisfving the requirements of the Method of Difference. We obviously cannot find cases in which gravitational influence is absent.

Method of Concomitant Variation.—Another expedient selected by Mill as available when the Method of Difference is not applicable is that of the observation of concomitant variations. Its utility lies principally in its applicability to circumstances in which some circumstance cannot be wholly removed so as to produce the requisite negative instance, but can only be altered in amount. The Principle is formulated in

The Fifth Canon. (Method of Concomitant Variations.)

Whatever phenomenon varies in any manner whenever another phenomenon varies in some particular manner, is either a cause or an effect of that phenomenon, or is connected with it through some fact of causation.

An example of the use of this method is afforded by a study of the influence of the moon upon the tides. We cannot eliminate the influence of the moon altogether so as to obtain one case of its 'presence' and another of its 'absence'. In Mill's language the proximity of the moon is a 'permanent cause'.

We can, however, make simultaneous observation of variation in the position of the moon and of the times and places of high water. In this way we obtain data conforming to the following schema:

$A'BC \;.\;.\;.\; a'bc.$
$A''BC \;.\;.\;.\; a''bc.$
$A'''BC \;.\;.\;.\; a'''bc.$

Here, A′, A″, A‴ represent variations in the position of the moon; *a*′, *a*″, *a*‴ represent either corresponding variations in the level of water at a fixed place or corresponding variations in the places when the tide is at its highest. From such data causal connection may be inferred.

The principle is important, more important than Mill realized. In fact it is convenient to distinguish Inductive Methods according as they employ data relating to the presence and absence of phenomena or data concerning the variation of phenomena.

In the present chapter attention will be restricted to the former case. The latter leads to certain developments which will require separate discussion.

§ 3. Special Presuppositions of the Methods of Agreement.

In his exposition of the experimental methods Mill adopted, to begin with, an extremely simple view concerning the general course of nature, a view that was progressively modified in the light of the inevitable complications which emerged in the development of his theory. To do justice to his doctrine it will be convenient to adopt a similar procedure—to commence with a patent over-simplification, and then to consider the complications which require to be introduced. This procedure is of especial advantage in connection with the two Methods of Agreement, the method so described by Mill and the so-called Joint Method, which latter we have seen to consist in a double application of the principle of agreement. Without some special assumption either as to the constitution of nature as a whole or of the field to which these methods are applied they are devoid of any serious claim to logical cogency.

The course of nature would seem to have been regarded by Mill as a plurality or succession of discrete and unitary ‘phenomena’—to employ a conveniently vague and comprehensive term for the items in the field of observation and experiment. Each ‘phenomenon’ has its antecedents, its concomitants and its consequents. To some of these it stands in invariable relations. It is presumed, moreover, that at the outset of the inquiry the possibly relevant items have been fully ascertained. It remains only to discover which is related to which.

The field of investigation might thus be represented in some such way as the following:

. . .	*s*	*a*	*s*	*t*	*c*	*k*	*e*	*d*	*n*	. . .
. . .	*t*	*n*	*m*	*d*	*a*	*s*	*c*	*k*	*m*	. . .
. . .	*b*	*c*	*k*	*p*	*o*	*t*	*r*	*l*	*q*	. . .
. . .	*d*	*e*	*r*	*q*	*l*	*n*	*q*	*b*	*o*	. . .
. . .	*q*	*o*	*l*	*e*	*b*	*m*	*p*	*a*	*t*	. . .

Here, each column of letters represents a group of concurrent phenomena, or a general state of affairs at a certain time, any item of which may be selected as a subject of investigation. The successive columns represent successive states of affairs. It is assumed that each individual phenomenon has its cause somewhere among the immediate antecedents and that each phenomenon is caused in only one way. Which antecedent is the cause of any given phenomenon remains to be ascertained.

Under such conditions the procedure is extremely simple. Demonstrative certainty is secured merely by the Method of Agreement.

Let us suppose that we wish to ascertain the cause of *r*. From the circumstances of its first manifestation we know only that the cause lies among the antecedents *a*, *n*, *c*, *e*, *o*. But comparing the first and second instance we obtain the following data:

a **n** *c* *e* *o* . . . *r*.
k *s* *t* **n** *m* . . . *r*.

With these two instances before us we may reason, as Mill says, in the following way: The antecedents *a*, *c*, *e*, and *o* cannot be the cause of *r* since each is absent in the second case when *r* is present. Nor can the antecedents *k*, *s*, *t* and *m* be the cause since they are absent in the first case. The only invariable antecedent is *n*. Hence this must be the cause. The argument can be cast into the form of demonstrative induction:

Major premiss: If any antecedent is alone common to two or more instances of a phenomenon that antecedent is the cause of that phenomenon (i.e. is its invariable antecedent).

Minor premiss: The antecedent *n* is alone common to two cases of *r*.

Conclusion: *n* is the cause of *r*.

The major premiss is an immediate inference from the assumption that every event has a cause where 'a cause' is defined as an invariable antecedent. The minor premiss is provided by observation. The conclusion is a specification of the major premiss and a generalization of the minor.

The same general schema enables us to give a simple interpretation to the so-called 'Joint Method'. This method as we have seen consists not in a combination of the Methods of Agreement and Difference but in a double application of the Method of Agreement. It depends upon the conjunction of a set of cases which agree in the presence of one phenomenon only with a set of cases agreeing only in the absence of that phenomenon.

The Method of Agreement requires the principle: *Whatever circumstance is alone present when the phenomenon is present is the cause of the phenomenon.* To this Mill adds the supplementary principle: *Whatever circumstance is alone absent when the phenomenon is absent is the cause of that phenomenon.* The combination of these two principles constitutes the basis of the Joint Method. Thus let us suppose that we are interested in the phenomenon *n*. Comparing two instances, say those of columns 2 and 6, we obtain the following data:

$$\begin{array}{cccccccccc} s & t & \mathbf{b} & d & q \ldots & a & \mathbf{n} & c & e & o. \\ c & a & o & l & \mathbf{b} \ldots & k & s & t & \mathbf{n} & m. \end{array}$$

From this we should infer, in accordance with the positive principle of agreement, that *b* is the cause of *n*.

Comparing next the three situations represented by columns 3, 4 and 5 we find that they do not agree in respect of the presence of any phenomenon whatever. Nor do they agree in respect of the absence of any phenomenon, except in one respect. Within the range of the phenomena considered, they agree in the absence of *n*, and in that alone. This provides us with the requirement of the negative principle of agreement. Representing the absent phenomena by the symbols $\bar{a}$, $\bar{b}$, $\bar{c}$, etc., and rearranging their order to facilitate comparison, the data are as follows:

Antecedents.	Consequents.
$a\overline{\mathbf{b}}c\overline{d}ek\overline{lmn}op\overline{qrst}$	$\overline{abcde}klm\mathbf{n}\overline{opqr}s\overline{t}$
$\overline{a\mathbf{b}cde}klmn\overline{opq}rs\overline{t}$	$\overline{ab}cde\overline{klm\mathbf{n}o}pqr\overline{s}t$
$\overline{a\mathbf{b}c}de\overline{klmno}pqr\overline{st}$	$abc\overline{de}kl\overline{m\mathbf{n}}op\overline{qrst}$

Here the only common antecedent is $\bar{b}$ and the only common consequent is $\bar{n}$. Hence in accordance with the negative principle of agreement we infer that b is the cause of n.

The survival, both in Mill's system and in subsequent works, of the Method of Agreement and the Joint Method, notwithstanding their special and dubious presuppositions, must be attributed to the fact that after every criticism has been made they retain a certain residuum of value. They may be defended on two grounds. It may be contended, firstly, that they provide rough and ready methods of probable reasoning in certain departments of research. Alternatively, it may be argued that their special requirements are satisfied under certain definable conditions, and that under these conditions they yield demonstrative conclusions.

It is plausible to suppose that the course of nature is such that when the antecedents of any phenomenon conform to the requirements of the two canons of agreement causal connection may be problematically inferred. It is on this ground that the methods are commonly defended. The difficulty is that the probability obtained is wholly indeterminate. In this respect the Method of Agreement stands on the same footing as the procedure of simple enumeration from which, in fact, it can scarcely be distinguished. More importance attaches to the Joint Method, since whatever probability is reached by the application of the simpler method of agreement is considerably enhanced by the evidence of the negative cases. But, even so, the probability remains indeterminate.

A defence of these methods as forms of demonstrative reasoning may be based upon the fact that under certain exceptional conditions their peculiar requirements are sufficiently provided for. This is notably the case when a body of more or less systematic prior evidence renders it highly probable that the cause factor to be ascertained is one, and only one, of a certain enumerable set of items. Examples of this are found even in the backward sciences. Within the province of 'psychical research', there is some evidence that an essential condition of 'the poltergeist phenomenon' lies in the presence, somewhere in the haunted building, of some individual who is either a 'medium' or possessed of neuropathic traits. It may be required to ascertain, in a given case, who this individual is. If we find in such a haunted house, occupied by five persons, A, B, C, D and E, that the haunting occurs when only A, B and C are present, and again when only A, D and E are present, we may infer that the presence of A is a determining condition of the phenomenon. Here the conclusion is demonstrative in the sense that it acquires the same measure of probability as attaches to the original hypothesis, viz. that the cause resides in A or B or C or D or E.

Important cases are provided in more advanced sciences wherever there is reason to suspect parallel causal series of events. This type of parallelism is exemplified by the process through which a stimulus gives rise to a sensation. This process consists in a sequence from physical stimuli, via the sense-organs and special neural paths to some distinguishable area of the brain. Physiological activity in this area is correlated with the occurrence of a sensation. In this way the exercise of each of the sensory functions is linked to a specific area of the cerebral cortex. Sight it is suggested is 'localized' in one area, hearing in another, smell in another, and so on. Within the limits of this hypothesis we may wish to ascertain which area of the cortex is so related to a given function - let us say, visual perception. The method of agreement may be of special importance here in view of the difficulty of experimenting with the human subject. The data are provided by cases of cerebral injury associated with loss or impairment of mental functions. Mapping the cerebral cortex into natural or artificial segments represented by the letters, A, B, C, . . . N, the data provided by three cases might be expressed in the following way:

Case 1. Affected areas: A H K L.
Case 2. ,, ,, H B D K M.
Case 3. ,, ,, C F H L.

Since H is the only area affected in common by all the cases of impaired visual function (so far as this is due to cortical injury), the conclusion is drawn that H is the relevant area for sight.

It would be technically correct to apply also the Joint Method under these conditions, but to do so would be impracticable, and, if not impracticable, logically redundant. It would be impracticable because it is more difficult to find a set of cases of brain injury which agree only in the absence of injury to a single area. It would be logically redundant since on the hypothesis in question the positive method is conclusive in itself.

It is only when the conclusion is problematic that the combination of the positive with the negative procedure has greater force than the use of one alone. The importance of the Joint Method lies in the negative method which it contains. In some cases, perhaps, we may find only negative agreement where positive agreement is less readily obtained.

The bonds which connect sensory experience with physiological activity in certain regions in the cortex would seem to be but the terminal phase of a process in which a fairly thoroughgoing parallelism is to be observed—a process which connects the appearance of sensory qualities with physical stimuli. Consequently the same general procedure is available for ascertaining the connection between the final and the initial stages of the process; or between any two stages whatever. In this way, for example, we may ascertain that the intensity of a sound depends upon the amplitude of the wave which stimulates the ear, and that the pitch is determined by its frequency.

In this connection it is of more than historical interest to recall that the inductive methods of Mill's precursor, Francis Bacon, were designed with reference to a closely analogous problem. Bacon had supposed that the chief problem of scientific inquiry was to ascertain the particular forms of corpuscular structures upon which the qualities of things depend. A 'form' is such that given this form a certain quality invariably follows, and if the form is absent then this quality is absent. Now, provided we know that the number of forms a body may assume is limited, it becomes a simple problem to discover which specific form determines which specific quality. In this way, for example, we may discover that heat depends upon motion.

Mill, however, in adopting Bacon's scientific methods discarded the underlying theory. He endeavoured to extend the range of their application to the discovery of causes of every kind. This was in many respects an advance in inductive theory, but at the same time it exposed Mill to objections from two points of view. His methods were open to objections directed to their claims of practical utility and they were open to objections on the score of their logical cogency.

It is evident that even if the course of nature had, quite generally, the character which we have provisionally supposed, the methods would be wholly useless if either the antecedents were incompletely known or if they exceeded a certain measure of complexity. In point of fact both sources of difficulty arise. The general condition that all the antecedents of a phenomenon are ascertained and separately discriminated is one that is never even approximately realized. Many phenomena, through their microscopic character or for some other reason, are beyond our powers of observation even when we have at our disposal instrumental aids. Moreover, even within our limitations, any column in our diagram which represents a given state of affairs must be indefinitely prolonged. We are thus impaled upon the horns of a dilemma. If our knowledge of the antecedents of a phenomenon is restricted we can never know that in any given set of cases there is only a single similarity or difference. On the other hand, in proportion as our knowledge is extended, the less probable it becomes that the cases will agree or differ in a single particular alone. In either case we are debarred from reaching any reliable conclusion.

The force of such considerations becomes apparent if we consider any case in detail. We are interested, let us suppose, in the cause of the obscure phenomenon of 'table turning', cases of which are provided at meetings of 'spiritualists'. What are the antecedents? Are we to take into account all the physical and chemical properties of the table and of all the material surroundings? Are we to record all the physiological and psychological states of the participants in the occasion? Are we, above all, to treat seriously the alleged states of supersensible beings in a supersensible world? Such questions have only to be asked in order to reveal the impotence of the experimental methods in any such inquiry.

Formidable as such objections might appear Mill has, nevertheless, at his disposal a wholly adequate reply. There is nothing in his theory to commit us to the view that if the methods are inapplicable anywhere then they are useless everywhere. On the contrary, Mill's own account suggests that their application must be restricted to cases in which there is some approximation to full knowledge of the possibly relevant conditions or to cases in which a considerable number of circumstances can be systematically ignored on account of their known irrelevance.

The objections which may be raised against the logical cogency of the methods are more serious. Of these, too, Mill was in great measure, though not entirely, aware. It is obvious, at least, that we must dispense with the assumption that every phenomenon is determined in only one way. We must admit, that is to say, ' plurality of causes '.

We have next to eliminate the over-simplification involved in the supposition that the cause is ever simple. A simple phenomenon *p* is never the effect of a correspondingly simple antecedent representable as *a* or *b* or *c*, but depends upon a conjunction of antecedents such as *abc*.

Finally, we may note the complications which arise when we recognize, as Mill himself recognized, but inadequately, the fact that phenomena not only differ in respect of presence and absence but also vary in degree. This last consideration raises questions of a very special kind. These will occupy us later. Our immediate concern will be to inquire how causal factors may be ascertained given (1) that a total cause is complex, and (2) that a phenomenon may be caused in several ways. For this purpose two methods are of fundamental importance, Mill's Method of Difference and a modified version of his Method of Agreement.

§ 4. Agreement and Difference.

In place of the simple schema of page 298, we have to contemplate a world in which any phenomenon *p* stands in relation to its conditions in a manner represented in the following way :

$$\begin{array}{llll} (a & b & c) & \\ (f & g & h) & \qquad p. \\ (m & n & o) & \end{array}$$

That is to say, any phenomenon, *p*, may be caused either by some conjunction *abc*, or by a conjunction *fgh*, or by a conjunction *mno*, etc. We cannot limit the possibilities in advance.

With such general relations presupposed, the inductive logician may proceed to formulate methods appropriate to the discovery of causes and cause-factors, understanding by the latter term any constituent such as *a*, *b* or *c* in the complete cause, *abc*. But it is clear that the process of ascertaining ' complete causes ' must be subsequent to the discovery of individual cause-factors. Given that a certain complex situation, *abcdef* contains one such complete cause, the procedure in general will be to show that some of these constituents, say, a, b and d, are causally operative whereas the remainder, c, e and f, are causally irrelevant. The problem, therefore, resolves itself into the investigation of individual cause-factors—with regard to which the following special cases require to be considered:

(1) That in which a circumstance may be shown to be a factor in *every* cause of the phenomenon. A factor of this kind might be described as an absolutely necessary condition of the phenomenon. Alternatively we may say that, in this case, the phenomenon is absolutely dependent on this factor.

(2) That in which a circumstance may be shown to be a factor in *no* cause of the phenomenon, i.e. in which the phenomenon is absolutely independent of this circumstance.

(3) That in which a circumstance is shown to be sometimes, but not always, a factor in the production of the phenomenon.

Alternatively, the problem may be stated simply as that of ascertaining whether a given circumstance is causally relevant or irrelevant to a phenomenon, and whether its relevance or irrelevance is partial or complete. It is by their bearing upon problems of this kind that methods such as Mill's must be judged.

The Method of Agreement, as Mill himself saw, is vitiated by the possibility of alternative causes. Apart from supplementary information, specifying very exceptional conditions, this method is incapable of providing an answer to any of the questions we have raised. Suppose, for example, that we wish to determine the cause of success and happiness exhibited in the lives of two men, Jones and Smith. The relevant circumstances are as follows: a, the inherited wealth of Jones, as contrasted with $\bar{a}$, the initial poverty of Smith; b, the business acumen of Smith as contrasted with $\bar{b}$, its relative absence in Jones; and c, the particular constellation of stars under which, as it happened, both men were born. Being the sole common circumstance, c would be presumed by the method of agreement to be the cause of the phenomenon under investigation; whereas the probable alternative causes, a and b, would be eliminated as causally irrelevant.

Nevertheless, the proof of causal irrelevance may be provided for by a simple modification of this method. Mill had assumed that when two instances of a phenomenon agree in only one respect causal irrelevance is simultaneously proved with regard to every circumstance in which the two cases differ. This is clearly a mistake. Where there are two or more circumstances which are changed, as in the above example, they may be alternative causes. What is required is the progressive elimination of one circumstance at a time. The general principle may be stated in what may be called *The Principle of Independence:*

Any circumstance in which alone two instances of a given phenomenon differ is causally irrelevant to that phenomenon.

The case must thus conform to the following schema:

$$\begin{array}{l} a\ b\ c\ .\ .\ \ p. \\ a\ b\ \bar{c}\ .\ .\ .\ p. \end{array}$$

From this we may infer that c is causally irrelevant to p or that p is independent of c. But even this statement requires qualification.

Independence, we have noted, may be partial or complete. But it is clear that the two instances selected in accordance with the principle stated above are not sufficient to prove that independence is complete. Though p is independent of c when a and b are present it may yet depend upon c in the absence of a or b. Removing every stone in turn, we might show that the stability of a building is independent of each of its foundations taken in isolation; but it would also be true that the removal of any stone whatever might be followed by collapse in the absence of some selection from the rest.

Our principle therefore proves only partial independence; and of this our knowledge may be more or less determinate. The limiting case is that in which we know merely that, in the presence of *some* (unspecified) condition, p is independent of a given circumstance c. This is, in general, a trivial conclusion, equivalent merely to the assertion that p is not absolutely dependent on c. Such knowledge only acquires importance when it contradicts a prevalent error, or when there is some presumption that c is at any rate one of the alternative conditions upon which p depends.[1] The conclusion becomes more determinate and in consequence of greater importance in proportion as we are able to define with precision the circumstances under which *c* is causally irrelevant, i.e. in so far as it assumes the form:

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[1] A case of this kind is provided by the researches of K. S. Lashley, who showed, contrary to the prevalent view, that rats could learn to

Given *ab . . . n*, *p* is independent of *c*. Further refinement consists in restricting the range of *ab . . . n* subject to which the independence of *c* holds; and more particularly in determining upon which of the constituents of *ab . . . n*, *p* does in fact depend. The latter procedure requires the method of difference.

As causal irrelevance requires the principle of Independence so causal relevance requires a principle of Dependence. This may be stated:

Any circumstance in which alone a case of the presence of a given phenomenon differs from a case of its absence is causally relevant to that phenomenon.

This is the principle of Mill's Method of Difference, and requires instances conforming to the schema:

$$\begin{array}{lll} a \; b \; c & & p. \\ a \; b \; \bar{c} & & \bar{p}. \end{array}$$

As independence may be partial or complete so, too, with dependence; and as the principle of independence proves only partial independence, so the method of difference proves only partial dependence. If a case of the presence of *p* and one of the absence of *p* differ in their antecedents only in the corresponding presence and absence of *c*, we may conclude that *c* is a constituent of *one* collocation of circumstances upon which *p* depends.

But this, too, is a trivial conclusion. There is nothing paradoxical in the suggestion that in *some* circumstances any phenomenon may be a completing factor in the cause of any other. It is easy to imagine circumstances in which the dropping of a pin might issue in a social revolution; and, proverbially, a single straw may under certain circumstances break a camel's back. An infinite variety of alternative causes are conceivable.

thread mazes even after a considerable portion of the cerebral hemispheres had been removed. The conclusion here is that a condition presumed to be absolutely necessary is shown to be at most only necessary under certain conditions, or possibly wholly irrelevant. See K. S. Lashley, 'Brain Mechanisms and Intelligence', Chicago, 1929.

We require, therefore, to know something about the conditions of which c is the completing factor in the cause of p. If we know that these circumstances are exceptional and peculiar the fact that c is relevant to p is of less importance than if these circumstances are normal and of a kind which frequently occur. Commonly, any marked peculiarity in the circumstance will hardly escape notice. When the last straw proves fatal it is unlikely that we should fail to notice that the camel is already heavily burdened. But abnormality in the attendant circumstances may sometimes be concealed.

In any case, more important than knowledge that the conditions are commonplace or that they are abnormal is knowledge of *what the conditions are.* It would seem, then, that the important conclusions to be drawn by the use of the 'methods' are such as require at least three terms. The conclusion is not that p is dependent or independent of c, but that p is dependent or independent of c under conditions more or less defined. We may, therefore, restate the two fundamental methods in a manner which makes this consideration explicit. Let v stand for the circumstance which is varied, it being understood that only one varies at a time. Let C stand for a complex of conditions which are constant and adequately defined. Different variable conditions may be represented as v_1, v_2, v_3 . . . etc. Different collocations of constant circumstances by C_1, C_2, C_3 . . . etc. We then obtain two important cases.

I. In accordance with the principle of independence

$$Cv_1 \ldots p.$$
$$C\bar{v}_1 \ldots p.$$

∴ under circumstance C, p is independent of v_1.

II. In accordance with the principle of dependence

$$Cv_2 \ldots p.$$
$$C\bar{v}_2 \ldots \bar{p}.$$

∴ under circumstance C, p is dependent on v_2.

The fundamental methods therefore establish only partial dependence and partial independence, or rather dependence or independence which is not known to be more than partial. It remains to inquire what more can be ascertained.

Special importance attaches to the combination of the two conclusions in relation to precisely the same variant. Obviously we cannot obtain instances by which p is shown to be both dependent and independent of v relatively to the same constant collocation of circumstances ;[1] but it is not infrequent to find instances of the following kind :

$$\begin{array}{ll} C_1 v_1 \ldots p, & C_2 v_1 \ldots p, \\ C_1 \bar{v}_1 \ldots p, & C_2 \bar{v}_1 \ldots \bar{p}, \end{array}$$

i.e. in which p is shown to be independent of v_1 under one set of circumstances, C_1, and dependent upon v_1 of another set of circumstances, C_2. The discovery of four such instances would constitute the joint use of the Principle of Independence and the Principle of Dependence.[2]

Its importance lies in the fact that it establishes the fact that p is dependent upon but not absolutely dependent upon v_1. The instances required by the principle of dependence show that p is dependent upon v_1 and that there is no alternative condition to it within the complex of circumstances C_2. The cases conforming to the principle of independence prove that there is some alternative to v_1 within the circumstances C_1.

The proof of absolute dependence or of absolute independence is less readily obtained. In fact, it has already been observed that circumstances are conceivable in which anything might depend upon anything else. So far, therefore, it would seem impossible to prove absolute independence. It is similarly conceivable that every phenomenon may be produced in more than one way. If this be so then absolute dependence is beyond the range of demonstration.

The most that we can say is that the greater the variety of constant circumstances in which p is shown to be independent of v, provided there is no case in which it is shown to be dependent, the greater the probability that p is absolutely independent of v. Conversely the greater the variety of circumstances in which p is shown to be dependent on v, provided there is no

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[1] At least we cannot do so on the assumption of determinism. Three instances conforming to the schema,

$$\begin{array}{c} Cv_1 \ldots p. \\ C\bar{v}_1 \ldots p, \; Cv_1 \ldots \bar{p}, \end{array}$$

would of course constitute a proof that p is undetermined.

[2] This must, of course, be carefully distinguished from Mill's so-called Joint Method.

case of independence, the greater the probability that p is absolutely dependent on v. Whether there are any cases of absolute dependence or independence is a matter of small importance. These relations may perhaps best be regarded as limiting cases in a world in which everything stands in varying degrees of dependence or independence in relation to everything else.

§ 5. Mill's Methods: Summary and Conclusion.

In the present chapter we have been concerned, firstly, with the general principles of indirect induction, more particularly in so far as they are implicitly contained in Mill's doctrine concerning the 'Experimental Methods'; and, secondly, with the utility of such methods considered in themselves. In the light of this discussion we may attempt to answer the following questions:

1. To what extent, and under what conditions, does the employment of methods such as those formulated by Mill enable us to apply the 'Law of Universal Causation' so as to establish particular causal laws?
2. To what extent, and under what conditions, are Methods such as those of Mill of practical utility and logically cogent, irrespective of the assumption of this general law?

It has been seen that the most plausible form in which the principle of universal causation may be stated in accordance with Mill's analysis is the following:

For any event p there is a set of antecedents $v_1, v_2, v_3, \dots v_n$ such that

If $(v_1v_2 \dots v_l)$ or $(v_3v_4 \dots v_m)$ or $(v_5v_6 \dots v_n)$ then p.

If not-$(v_1v_2 \dots v_l)$ and not-$(v_3v_4 \dots v_m)$ and not-$(v_5v_6 \dots v_n)$ then not-p.

In this formula $v_1v_2 \dots$ etc., represent *undetermined* factors. Not-$(v_1v_2 \dots v_l)$ must be understood, of course, to stand for any collocation of which one or more of the components v_1 or v_2 or v_3, etc., are missing. The problem which arises in any particular investigation is to identify these unknown antecedents as $a, b, c, \dots$ etc. For this purpose we have seen two principles are available:

(1) The Principle of Dependence (Mill's Method of Difference), which proves that a given antecedent is a member of at least *one* such collocation.

(2) The Principle of Independence (replacing Mill's Method of Agreement), which proves that a given antecedent is not a member of at least one such collocation.

The conclusions directly obtained by the application of these principles we have found to be trivial, inasmuch as there is nothing improbable in the suppositions, firstly, that *any* circumstance may in some or other context be causally relevant to any phenomenon whatever; and, secondly, that there is *no* circumstance on which any phenomenon whatever is absolutely dependent.

These principles, however, acquire importance when the conclusion yielded is of the form: p is dependent upon v, or independent of v, under the constant condition C; and the more precisely we define C the more important is this conclusion. Most important of all is the case in which C admits of complete analysis.

Clearly, complete analysis is only possible when C comprises only a restricted number of independent components. In consequence, the case is one in which the inductive investigation presupposes not only the general law of causation, but more specifically the knowledge that the relevant antecedents of p are contained within a given enumeration *abc* . . . *n*. Hence, we may say, that given,

(1) That p has some cause;

(2) That the cause factors are contained in *abc* . . . *n*, conclusions of the following kinds may be demonstratively established:

(1) That some of these circumstances are wholly irrelevant to p.

(2) That some are absolutely necessary to p.

(3) That some are relevant to p when combined with some of the remaining antecedents, and irrelevant when combined with others.

That p is absolutely independent of a given factor v is established by showing that its removal from any collocation upon which p is consequent does not involve the removal of p. That p is absolutely dependent on v is established by showing that it is a component of every collocation upon which *p* is consequent and that its removal in any of these cases involves the removal of *p*. That *p* is sometimes, but not always, dependent upon *v* is established by showing that there is at least one case in which the absence of *v* does involve the absence of *p* and at least one case in which the absence of *v* does not involve the absence of *p*.

It is in this way that specific causal laws may be demonstratively established by the procedure of indirect induction involving methods more or less similar to those formulated by Mill. But it is important to observe that the validity and the practical utility of this procedure depends not so much upon the principle of Universal causation as upon the more specific major premiss that a given phenomenon *p* depends upon conditions within the enumeration *abc . . . n*. Demonstrative induction is a possible procedure whenever we are concerned with a phenomenon which is known to be determined. *Universal* causation needs to be postulated only in justification of the contention that the methods are universally applicable.

It is obvious, however, that the conditions required for demonstrative conclusions are not easily secured. The knowledge that the conditions of any given phenomenon lie among a specified group of antecedents is always highly problematic. It is always the product of some previous induction. This is one of the most serious limitations of the methods. Another serious limitation is that which arises from their restriction to cases in which the phenomenon and its antecedents are merely present or absent, as opposed to the cases in which they vary in ' degree '. The latter will occupy our attention in the following chapter.

BIBLIOGRAPHICAL NOTES.

Chapter XV.

Mill's own exposition of the Experimental Methods is to be found in Book III. of **The System of Logic,** Chapters VII., VIII. and IX. Chapter VII., though frequently neglected, is of importance in that it reveals some of the presuppositions of these methods.

In a historical study of the conception of formal scientific methods the student should acquaint himself with Bacon's treatment of the subject (**Novum Organum**) and with the treatment in **Herschel, Natural Philosophy.** The topic is also discussed briefly by **Hume, Treatise of Human Nature,** Book I., Part III., Section XV.

Almost every important work on inductive Logic, since Mill, contains a discussion expository or critical of the methods. **Venn's Empirical Logic** may still be read with profit in this connection. Amongst the more radical of recent discussions is that of **Johnson** (**Logic**, Part II., Chapter X.). Though in intention sympathetic to Mill, Johnson here reveals the limitations of the methods more effectively than do many of Mill's less indulgent critics. The discussion is of especial interest in that it embodies the first attempt to distinguish with precision between what is genuinely demonstrative and what is merely problematic in inductive procedure. In consequence Mill's Methods are transmuted into the Four Figures of Demonstrative Induction to be discussed in the following chapter.

The critical exposition in **Stebbing, Modern Introduction to Logic** (Chapter XVII.) provides another important contribution to the subject.

CHAPTER XVI.

UNIVERSALIZATION AND FUNCTIONAL DEPENDENCE.

§ 1. Determinate and Indeterminate Generalizations.

REFERENCE has been made in previous chapters to the distinction between absolutely determinate characters and the determinables under which these determinates fall; but we have had no occasion, so far, to employ this distinction in relation to the problems of induction. In common with most exponents of inductive theory we have proceeded as though there were no difference of principle according as a generalization concerned characters of a determinate or of an indeterminate kind. This assumption now requires correction. Consider the proposition: *Lightning is always followed by thunder.* Four cases may be distinguished on the basis of differences in the degree of determinateness in the characters involved. The cases are:

(1) That in which the subject or antecedent, 'lightning', is determinately defined; and in which the predicate or consequent is relatively indeterminate;
(2) That in which the antecedent is indeterminate and the consequent is determinately defined;
(3) That in which both are relatively indeterminate;
(4) That in which both are quite determinately defined.

In the first case what is asserted is that a specific sort of lightning is always followed by some or other sort of thunder. In the second case it is asserted that every sort of lightning is associated with some (the same) specific sort of thunder. In the third case we are asserting that every sort of lightning is followed by some or other sort of thunder, and in the fourth that a specific sort of lightning is always followed by a specific sort of thunder. Clearly the evidential requirements differ in the four cases. We might anticipate that the easiest generalization to establish would be the first. It is relatively narrow in its scope, and it is antecedently more likely that a given species of lightning should be followed by some or other kind of thunder than that the thunder should be always precisely of the same kind. The third case is similar to the first. The generalization differs only in its scope. What, in the first case, requires to be proved with regard to a specific sort of lightning has here to be proved concerning every sort. In the second and fourth cases determinateness is required in the predicate or consequent. Of these two cases the latter would seem to be the easier to establish. It is more likely that some specific sort of lightning will be followed by a specific kind of thunder than that every sort of lightning is.

From such considerations we may extract two general principles. Relatively to any body of evidence,

I. *The more determinate the subject of the conclusion the greater the probability of this conclusion.*

II. *The more indeterminate the predicate of the conclusion the greater the probability of this conclusion.*

We may next consider the bearing upon the conclusion of the degree of determinateness in the premisses. The conclusion, let us suppose, is the proposition *All men are mortal.* Here, as before, we may distinguish four cases. The terms 'men' and 'mortal' express respectively a relatively indeterminate subject and a relatively indeterminate predicate. A relatively determinate subject would be illustrated by the term 'Englishmen', and a relatively determinate predicate by a specification of the limits of mortality, say, 'dying before the age of 120 years'. Our four possible conclusions would then be

(1) All Englishmen die sooner or later.
(2) All men die before the age of 120 years.
(3) All men die sooner or later.
(4) All Englishmen die before the age of 120 years.

We may obtain similarly four types of premisses:

(1) All examined and recorded cases of Englishmen (who were born before a certain date)[1] are known to have died sooner or later.

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[1] Some such restriction in the definition of the evidence is of course always necessary, since our contemporaries are included amongst those for whom mortality must be proved.

(2) All examined cases comprising men of all nations are known to have died before the age of 120 years.

(3) All examined cases comprising men of all nations are known to have died sooner or later.

(4) All examined cases of Englishmen are known to have died before the age of 120 years.

Since each type of premiss might be adduced for each type of conclusion sixteen cases arise. It is not necessary, however, to consider these in detail. It is evident that two general principles may again be extracted:

I. *The more indeterminate the subject of the premiss the higher the probability of the conclusion.*[1]

II. *The more determinate the predicate of the premiss the higher the probability of the conclusion.*

Combining these principles with those previously stated we may say that probability is enhanced by indeterminateness in the subject of the premiss or by determinateness in the subject of the conclusion, and by determinateness in the predicate of the premiss or by indeterminateness in the predicate of the conclusion.

§ 2. Parallel Laws.

In accordance with the principles stated in the previous section we might hope to strengthen inductive arguments by the simple expedient of restricting our observation to cases which may be determinately defined. For example, instead of endeavouring to establish some generalization concerning men in general we might attempt, first of all, to generalize with regard to a certain human type. Ultimately we might hope to reach a generalization concerning every man by proving it to be true for each type in turn. In general, this procedure is sound, but a difficulty arises if we carry the policy to extremes. For, clearly, the more determinately a 'phenomenon' is defined the smaller the number of cases which arise for observation. If we insist on absolute specificity we may find ourselves with only a single case to observe. Nature, perhaps, cannot provide us with two grains of sand which are absolutely alike, and no two men are precisely alike in all their characteristics.

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[1] It should be noted that in this formula indeterminateness in the subject does not mean that we are ignorant whether the men are all of the same kind or not, but that it is known that only a relatively indeterminate description will cover all the cases.

We might attempt to meet this difficulty by a compromise, by demanding the maximum degree of specificity consistent with the minimum requirements in respect of number. But as yet we have found no principle which enables us to define the number of cases sufficient for generalization. For a more radical solution of the problem we must return to one of Mill's neglected clues—the evidential value of a ' parallel law '.

Under the general descriptive title of ' indirect induction ' we have distinguished two ways in which one generalization may be strengthened by being related to others.

The first was that in which a narrower generalization *All A is P* may be established by means of an independent induction *All M is P*, when from the latter the former may be deduced. The second type of procedure was that in which *All A is P* is shown to be probable on the ground that *All B is P*, *All C is P*, etc., where in fact A, B, C, etc., are all cases of M but where the wider generalization *All M is P* cannot be asserted with assurance.

These two types of procedure might be illustrated from arguments concerning the unknown elements. All known elements are subject to gravitation. This wider generalization may be employed as we have seen both to establish the relatively indeterminate conclusion that a given newly-discovered element is subject to gravitation and, by the procedure of demonstrative induction, to determine some more specific gravitational property.

The same evidence, however, will also establish the same generalization in the absence of any intermediate conclusion concerning the elements in general. If a single substance had been found not to be subject to gravitation the general proposition concerning all elements whatever would be proved to be false. It would not, however, invalidate the corresponding inferences regarding a single newly-discovered element. It would still remain probable that this particular element was subject to gravitation. We may next consider the case in which the properties under investigation are more determinate. All elements are subject to gravitation; but they differ from one another in their more specific gravitational properties. They differ, for example, in atomic weight. Let A, B, C, D, etc., as before, stand for the different elements. B, we may suppose, has been shown to possess the specific atomic weight p_1; C has the weight p_2, D has the weight p_3. Similar determinations have been made for other elements. What can we infer with regard to A?

We might, of course, infer that A has probably some specific property of the same kind; but to be content with this inference alone would be to neglect much that is of evidential value in the data.

Where the determinates $p_1, p_2, p_3, \ldots$ under the determinable P admit of arrangement in a systematic order there is a certain presumption that some corresponding order will be found in the related items, A, B, C, D, ... Where $p_1, p_2, p_3, \ldots$ represent specific atomic weights and A, B, C, D, . . . the various elements defined in terms of other properties we find the anticipated order formulated in what is known as the Periodic Law. The elements fall into a series composed of definite groups, ordered by certain relations between their characteristic properties, so that differences in atomic weight can be more or less systematically paired with differences in other properties. The possibility of such formulations exemplifies an important development of the conception of parallel laws.

This conception admits of further refinement and generalization. The chemical properties in terms of which the different elements are distinguished are extremely varied and complex. The case is greatly simplified when we are able to select two series of specific properties which fall respectively under two determinables. It is then frequently possible to establish a series of parallel laws in each of which the presence of one determinate under one determinable is uniformly related to the presence of a corresponding determinate under the other determinable. Of this we find an illustration in the well-known experiments of Galileo through which it was established that the velocity of a falling body is proportional to the time through which it has been falling. The data are as follows:

In observed cases of bodies falling for the time t_1 the velocity attained was v_1.

In observed cases of bodies falling for the time t_2 the velocity attained was v_2.

In observed cases of bodies falling for the time t_3 the velocity attained was v_3.

Each set of observations provides the basis for the generalization concerning all cases in which the same determinate properties appear. The set of parallel laws so established provides the basis for a further inductive process the characteristics of which we may now proceed to examine.

§ 3. Universalization.

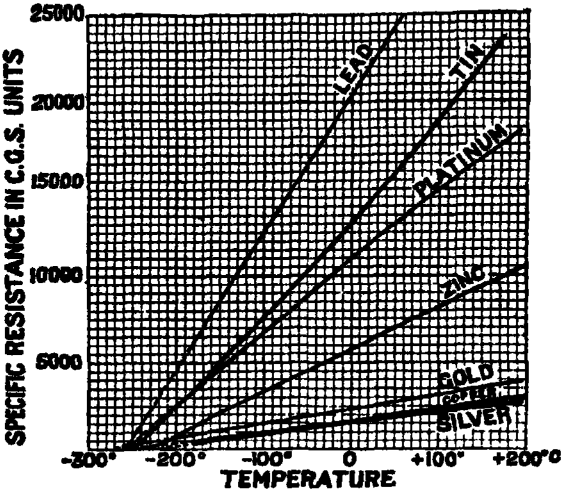
Everyday experience provides many examples in which variations in the specific characters under one determinable are linked with variations in the specific characters under another determinable, or in which variations in the properties of one thing are linked with variations in the same properties in another. Variations in the direction and velocity of the movement in a billiard ball may be matched with corresponding variations in the movement of the cue. A bar of metal passes through various shades of red and white as its temperature is raised. More precise correlations of this kind are provided by scientific research. Accurate measurement enables us to state with some precision how change in one variable is related to change in another. These relations are exhibited by the ordinary modes of graphical representation.

We have thus many occasions to inquire into the relations between a set of determinate properties p_1, p_2, p_3, . . . under a determinable P and a set of determinate properties q_1, q_2, q_3, . . . under another determinable, Q ; where preliminary observations suggest that p_1, is invariably associated with q_1, p_2 with q_2, p_3 with q_3, and so on for other values of P and Q.

In such cases the data may be conveniently presented in a table of the following form :

S_1 is q_1 when p_1	S_2 is q_1 when p_1	S_3 is q_1 when p_1	. . . S_m is q_1 when p_1
S_1 is q_2 when p_2	S_2 is q_2 when p_2	S_3 is q_2 when p_2	. . . S_m is q_2 when p_2
S_1 is q_3 when p_3	S_2 is q_3 when p_3	S_3 is q_3 when p_3	. . . S_m is q_3 when p_3
. . .	. . .	. . .	. . . .
S_1 is q_m when p_m	S_2 is q_m when p_m	S_3 is q_m when p_m	. . . S_m is q_m when p_m

The symbols S_1, S_2, S_3, S_m, here stand for particular *things ;* p_1, p_2, p_3, p_m, q_1, q_2, q_3, q_m, stand for determinate *properties.* Any line therefore stands for a set of numerically distinct things with precisely the same property, any column for numerically the same thing characterized (at different times) by different specific properties. Thus S_1, S_2, S_3, S_m, might stand for *m* pieces of platinum wire of precisely similar dimensions; p_1, p_2, p_3, p_m, for specific degrees of temperature, and q_1, q_2, q_3, q_m, for specific measures of electrical resistance. The data contained in any column are represented graphically in the line for platinum in the following figure. The data corresponding to the curves for other metals could be tabulated in the same way.



From evidence of this kind inference may proceed in two dimensions. One dimension is that in which from the instantial premiss

$$S_1, S_2, S_3, S_m \text{ are } q_1 \text{ when } p_1,$$

we infer

(1) *That a certain unexamined S, say S_n, is q_1 when p_1.*

(2) *That every S is q_1 when p_1.*

This is the familiar procedure of direct generalization. The second dimension of inference is illustrated by the case in which from the premiss

S_1 is q_1 when p_1, S_1 is q_2 when p_2, S_1 is q_3 when p_3 . . . S_1 is q_m when p_m,

we infer

(1) *That S_1 is q_n when p_n.*

(2) *That for all values of P which characterize S_1 there is some corresponding value of Q.*

This is the type of inductive procedure which we have described as 'Universalization'.[1]

Generalization is thus a process of inference in which from a set of examined instances *similar to one another* a conclusion is inferred concerning other instances *similar to those examined.* Universalization is a process of inference in which from a set of examined instances which *differ from one another* in a certain way a conclusion is inferred concerning others which *differ from these examined.* Generalization, in the extreme form, is concerned with a class of 'things' or 'particulars'; universalization, in its extreme form, is concerned with a class of properties or 'universals'. In both cases, of course, the inference may be highly problematic. In both, too, the inference may be said to depend upon the *number* of cases examined; generalization depends upon the number of particulars, universalization upon the number of specific properties. There is, however, this important difference. Within the scope of a generalization there is always a potentially infinite number of cases comprised; and of the total range of cases those examined will always constitute an insignificant fraction. The total range of determinates under a given determinable, on the other hand, may be definitely restricted and under favourable conditions the whole range may be open to inspection. In this way universalization may conform to the requirements of summary induction.

So far, we have considered only inference from the data of a single line and inference from the data of a single column. We may also base our inference upon a set of different lines and upon a set of different columns, and the two operations can be combined. On the basis of the data as a whole the conclusion reached may assume the following form:

For every S there is a value of Q for each value of P.

This conclusion may be regarded as a 'secondary' induction, i.e. as an induction from premisses which are themselves inductions. We may suppose the operation to consist either in a series of generalizations the conclusions of which are universalized or in a series of universalizations the conclusions of which are generalized. It is misleading, however, to treat the two operations as wholly independent. It has been seen that the existence of 'parallel laws' increases the probability of any given law. The probability that every S is q_1 if p_1 depends not solely on the instances of S that are both q_1 and p_1 but also upon the evidence from which we generalize to the effect that every S is q_2 if p_2, q_3 if p_3, etc. The probability of any generalization is enhanced by the possibility of universalization, and the probability of any particular case of universalization is enhanced by the possibility of generalization. Though the two operations mutually reinforce each other they preserve a certain partial independence. There are cases in which generalization is reasonable and universalization dubious. There are also cases in which the former is more hazardous than the latter. Generalization is more reasonable than universalization when there is a large number of similar cases, but only a small range of variation in the determinables involved. Thus we generalize freely that anyone who takes a large dose of poison will die, but we are not prepared to make any definite assertion concerning the effects of smaller doses of the same poison or similar doses of other substances of a slightly different kind. Universalization has priority when a smaller number of cases have been more intensively studied. Careful study of the effects of varying the dose of a given medicine in the case of a single patient might enable a doctor to predict the effects upon this patient of further variations in the dose, whereas he might be doubtful as to the effects of the same prescription if applied in another case.

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[1] Ch. XII., p. 261.

The difference between a generalization concerning a class of particulars and a generalization concerning a class of universals may be expressed as a difference in propositional form. It is the difference between general propositions of the form *All S is P* and assertions of functional dependence, i.e. assertions of the form $y = f(x)$. In a previous chapter we were concerned with the difference between the conception of a 'function' as it occurs in elementary mathematics and as it occurs in logic and more advanced branches of mathematics. In the present context we require to consider the use of this conception in the empirical sciences.

The conception of a 'variable' is fundamental. In any expression of the form $y = f(x)$, y and x must stand for any member of a certain *set*, *class* or *series*. It would be nonsense to say that the sum of the angles of a triangle is a function of the number of its sides, because neither of these terms are variables: the number of angles is always three and their sum always equal to two right angles. On the other hand, we may say that of the sum of the angles of a closed rectilineal figure is a function of the number of sides it contains, since a rectilineal figure may have 3, 4, . . . n sides, and as the number of sides increases so does the sum of its angles.

In the more familiar examples the variables are restricted to numbers, and the functional expression is correspondingly restricted by the sort of relation which may hold between numbers, viz. to the sort of relations which are exhibited as holding between two or more variables by an equation.

In the more extended sense of functional dependence the x's and y's may stand for any sort of thing whatever, since everything is a member of some set or series, and the f may stand for any manner in which the members of one set may be similarly related to the members of other sets.

We have now to consider the case in which x stands for any member of one set of characters and y for any member of another set. This is the common form assumed by assertions of functional dependence in the empirical sciences. When, for example, we say that the *volume* of a gas is a function of its temperature we mean that for any given quantity of gas (pressure being constant) there is a set of temperatures t_1, t_2, . . . t_n, and a set of volumes v_1, v_2 . . . v_n so related that for each of the temperatures there is an associated volume.

Such statements of functional dependence may incorporate some mathematical function, but usually only with some restriction. The mathematical function will be restricted by the nature and scope of the set of characters involved. If, for example, we compare the intensity of our sensations with the strength of the stimuli which evoke them it is evident that the former are in some way functionally dependent on the latter. Attempts to define the nature of this dependence have led to the suggestion that the intensity of the sensation is a logarithmic function of the stimulus. The formula proposed is $S = K \log I$. It is evident, however, that we cannot substitute any quantities we choose in the place of the variables without regard to the nature of the characters referred to. We cannot, for example, apply the formula in a manner which involves negative quantities, either as negative stimuli or as negative sensations.

Certain other distinctions between functions are of importance in the theory of inductive inference. A function may be one-valued or many-valued; and it may involve any number of variables, and it may be determinately or indeterminately defined.

A variable, y, is a one-valued function of a variable, x, when the relation between the set represented by x and that represented by y is such that for each value of x there is one and only one value of y A variable y is a many-valued function of x when the relation is such that for one value of x there are alternative values of y. This may or may not involve that for one value of y there are alternative values for x.

An important kind of function is that which is known as *periodic*. A periodic function is such that a variation in x in one direction involves a variation in y in the same direction but only up to a certain point, after which further variation in x involves a variation in y in the reverse direction. Consider, for example, the variations in the angles formed by the two hands of a clock with the lapse of time. Let $x_1, x_2, x_3 \dots x_n$ represent specific periods of time, and $y_1, y_2, y_3, \dots y_n$ the dimensions of the smaller angles formed by the two hands. Starting at noon we may write $x = 0$ and $y = 0$. Assuming for simplicity that the small hand remains stationary we obtain the following values. When $x = 5$ mins. $y = 30^\circ$, when $x = 15$ mins. $y = 90^\circ$, when $x = 30$ mins. $y = 180^\circ$. At this point the changes in y are reversed in direction, for $x = 45$, $y = 90$, for $x = 55$ mins. $y = 30^\circ$ and for $x = 60$ mins. $y = 0$, and the cycle is again repeated. Nature exhibits innumerable examples of these periodic functions. They will be found to be of great importance in connection with an inductive problem to be considered in a later section of this chapter.

We may often have good reason to suppose that one character is functionally dependent upon another without being able to assert what specific function it is. Our knowledge of the nature of any function may thus be more or less determinate, and it may be indeterminate in various ways. We may have reason to suppose that for each value of x there is only one value for y and yet be unable to assign any determinate value of y when given a determinate value of x. There is an infinite number of one-valued functions: $y = x^2$, $y = x^3$, $y = x^4 \dots y = \frac{1}{x}$ . . . etc.

A second form of indeterminateness in our knowledge of functional relations concerns the *scope* of the function. We may have very good reason for supposing that a certain functional relation holds between certain limits, but it may be highly doubtful whether it holds when the values of x are indefinitely increased.

It has already been noted that the probability of a generalization varies with the determinateness of its terms. Similar considerations apply to assertions of functional dependence. Relatively to a given corpus of evidence the three conclusions, *y is a certain specific function of x*, *y is a certain sort of function of x*, *y is some function of x*, rank in ascending order of probability. We obtain higher probabilities in proportion as we are satisfied by indeterminate conclusions. Conversely, the more determinate the assertion of functional dependence the more exacting are the evidential requirements.

§ 4. Special Cases.

For simplicity, the typical instantial premiss has been represented in the form S is q when p. This formulation, however, applies primarily only to the case in which the same thing, S, combines within itself the two properties p and q. Cases such as the following are of equally frequent occurrence: S_1 is p when S_2 is p; S_1 is q when S_2 is p; if S is p at t_1 then S is q at t_2. The first of these is the case in which the possession of a given character by one thing determines the possession of the same character by another thing—as, for example, when the movement of one body determines the movement of another. Usually such determination depends on the spatio-temporal relations of the two things so that more explicitly the formula would be: If S_1 is r to S_2 then S_1 is p when S_2 is p. The second case differs from the first only in the fact that the two properties differ. The third case is that in which the possession of one property at one time by a given thing determines the possession of a different property by the same thing at a later time. This schema applies to biological laws of development. These differences in the form of the proposition do not involve any difference in the principles of inference involved. There are, however, some cases which require special discussion.

In the cases so far considered the individual things, S_1, S_2, . . . S_n, have entered only as centres to which the characters p and q may be referred. Other characters which S_1, S_2, . . . S_n

may have possessed have been presumed to be irrelevant, so that the symbol S might have been taken to stand for a 'bare particular' devoid of intrinsic characterization. When, however, 'S' has its own distinctive connotation, when it stands for a certain sort of thing the other characters of which are not known to be irrelevant to the relations of p and q, further considerations arise.

It is obvious, in general, that whatever interpretation is placed upon S in the premisses must be retained in the conclusion. If S_1, S_2, S_3, . . . S_n, stand for the observed pieces of platinum the conclusion, whether generalized or universalized, must relate to platinum as well.

We may consider next the procedure in which the inductive process is developed in a third dimension, in which the different subjects are species of a certain genus. Here we may find two dimensions of universalization. We are concerned, say, with the properties of the different metals. The data here consist of observations of the following kind:

$$\begin{array}{lll} S_\alpha \text{ is } q_1 \text{ if } p_1 & S_\beta \text{ is } q_1 \text{ if } p_1 & S_\gamma \text{ is } q_1 \text{ if } p_1 \ldots \\ S_\alpha \text{ is } q_2 \text{ if } p_2 & S_\beta \text{ is } q_2 \text{ if } p_2 & S_\gamma \text{ is } q_2 \text{ if } p_2 \ldots \end{array}$$

where the subscripts, α, β, γ, . . . mark the differences such as those between the metals, gold, copper, lead and so forth.

Two cases require to be considered. The first is that in which the differences in the subjects do not affect the values to be assigned to P and Q. This is illustrated in the law concerning the relation between the velocity of a falling body and the time for which it falls. Let S_α, S_β, S_γ, etc., represent balls of equal weight but made from different metals, gold, silver, iron, and so forth; q_1, q_2, q_3, . . . represent the various velocities the balls attain in falling; and p_1, p_2, p_3, . . . represent variations in time. We obtain precisely similar values for P and Q in each of the separate columns. In such cases the difference between the substances is seen to be irrelevant to the law connecting P and Q. The subscripts may therefore be ignored.

The second case is that in which the values to be assigned to P and Q differ in the different columns. Let S_α, S_β, S_γ, again stand for different sorts of metals, but let P stand for varying degrees of temperature and Q for corresponding measures of electrical resistance. We find in every case that a change in the temperature of the metal involves a change in the resistance which it offers to the passage of an electrical current, but the measure of resistance corresponding to a given temperature will differ in the different cases. Here we can still proceed to universalization from the data of each column. In this way we reach the laws connecting temperature and specific resistance for different metals, some of which are illustrated in the figure on page 320. It is by this procedure that we obtain the type of generalization involved in determining the specific properties of some material such as its coefficient of expansion, or its specific resistance.

The second dimension of universalization is one in which, in principle at least, some induction might be made concerning differences between the laws connecting temperature and resistance according to the constitution of the metal involved. Since, however, the difference between one metal and another cannot be expressed as differences in the determinates under a single determinable no very precise conclusion can be drawn. The most that can be inferred is that metals other than those examined would probably exhibit different laws.

The case in which this second dimension of universalization is possible is that in which the relevant differences between the different subjects can be expressed as differences under a single determinable.

Let the instances be provided by a series of pieces of metal wire similar in material and cross-section, differing only in length. We again proceed to determine variations in electrical resistance consequent upon variations in temperature. Here, since the differences between the relevant particulars relate to the same determinable, viz. length, we may substitute for the symbols S_α, S_β, S_γ, the symbols Sl_1, Sl_2, Sl_3, . . . etc. Symbolizing variations in resistance by r_1, r_2, r_3, . . . and differences in temperature by t_1, t_2, t_3, . . ., our data can be represented as follows:

$$\begin{array}{ll}
Sl_1 \text{ is } r_1 \text{ when } t_1 & Sl_2 \text{ is } r_m \text{ when } t_1 \;\ldots \\
Sl_1 \text{ is } r_2 \text{ when } t_2 & Sl_2 \text{ is } r_n \text{ when } t_2 \;\ldots \\
Sl_1 \text{ is } r_3 \text{ when } t_3 & Sl_2 \text{ is } r_o \text{ when } t_3 \;\ldots \\
\ldots\ldots & \ldots\ldots\quad\ldots
\end{array}$$

The symbols r_m, r_n, r_o, . . . here indicate merely that values must be assigned to R in the second column different from those in column one.

From any single column we may derive a formula which defines the manner in which resistance varies with temperature, the length of wire being constant. Similarly, from any single line we derive a formula which defines how resistance varies with the length of the wire for any constant temperature.

At this point, however, it is easy to see that the data may also be expressed in the form :

S is r_1 when l_1 and t_1 S is r_m when l_2 and t_1 . . .
S is r_2 when l_1 and t_2 S is r_n when l_2 and t_2 . . .
. . . .

That is to say, we may detach the distinguishing feature from the subject and exhibit r as a function of two variables, t and l. The same general principle may be progressively applied to all the components in the connotation of S, however complex S may be. These components will then fall into two groups : (*a*) those that are irrelevant in the sense that their variation does not affect the variables with which we are concerned, and (*b*) those upon which the values of the variables are dependent. The former can be eliminated from the formula altogether. The latter may be incorporated in the formulæ as one of the terms of the general law. Sooner or later, by this procedure, a point is reached at which S is entirely divested of content. We then obtain a generalization expressing simply the functional dependence of certain properties upon others. From this formula all reference to the particulars has been eliminated. Thus, assuming for simplicity that the length and the temperature of the wire were the only variables upon which its resistance depends the first conclusion would be

$$r = f(lt),$$

i.e. resistance is a function of length and temperature.

But even where some special property of a substance cannot be ignored as irrelevant it is often possible to express this property by a ' constant ' as contrasted with the variables. If we were concerned, for example, with establishing a general law regarding the dependence of the resistance of a wire upon (i) the nature of the material of which it is composed, (ii) its length and (iii) its cross-section, we should proceed to collect data which could be cast into the same general form :

S_a is r_1 when l_1 and s_1.
S_a is r_2 when l_2 and s_1.

Three dimensions would here be required, one for difference of length, l_1, l_2, l_3, one for differences of cross-section, s_1, s_2, s_3, and one for differences in material, S_α, S_β, S_γ, . . . The differences between S_α, S_β, S_γ, etc., would not form a continuous series of determinates under a given determinable, but their effects upon the final values of R can be expressed in quantitative terms. The influence of the constitution of platinum would appear as one constant value, that of copper as another, and so on. Consequently we reach a formula of the form :$R = f(kls)$ where l and s are variables and k a constant, which differs for different metals. The specific formula is

$$R = k\frac{l}{s},$$

i.e. the resistance has a value which is obtained by dividing the length of the wire by its cross-section and multiplying the quotient obtained by a certain constant which is fixed for each metal. With further inquiry it might, of course, be possible to show that the different constants in the case of the different metals are determined by variations in some one determinable in terms of which the different metals may be described, e.g. some characteristic in atomic or molecular structure. In this case the various special formulæ for determining resistance would be assimilated into a more general formula of the form $R = f(KLS)$ in which K as well as L and S is a variable.

In the foregoing account it has been convenient to assume that a single property varies in only one way. In point of fact, however, we find that almost every property varies in several dimensions. Thus the properties exhibited in pure tones exhibit primarily two dimensions of variation—variation in pitch and variation in intensity. Some properties exhibit more than two dimensions, the definition of these is, of course, in every case a matter for the relevant special sciences. No logical complications in principle arise from this complexity. It follows, however, that whilst there cannot be more than one dimension of generalization there may be several dimensions in which universalization may occur.

The theory of generalization admits, as we have seen, of two main lines of development. Firstly, we may formulate the conditions under which the probability of the conclusion may be enhanced. The limiting case in this direction of development is that in which the argument assumes a strictly demonstrative form. Secondly, we may extend the theory so as to cover various degrees of generalization. In this way inductive theory is absorbed into the more general principles of statistical inference. The complete generalization is merely the limiting case in this general theory. The theory of universalization admits of analogous extension.

The inferences so far considered yield conclusions that are problematic and indeterminate. We may attempt to enhance the probability of the inference, and perhaps in certain cases we may reduce it to strictly demonstrative form. We may also attempt to express functional laws in a quite determinate formula. These two lines of development are the subject of the two succeeding sections of this chapter.

§ 5. Technical Methods of Universalization.

Mill's 'Method of Concomitant Variations', though of little more than historical interest, provides a convenient point of departure for the subject of the present section. Mill, as we have seen, developed his technical inductive methods in terms of the presence and absence of phenomena. The conception of variation in degree was introduced merely to meet the supposed difficulty arising from 'permanent causes'. In this Mill failed clearly to see that the method of concomitant variation introduced a new principle. Moreover, this new principle is very much more potent than that for which it was intended to provide a kind of inferior substitute.

Inasmuch as 'concomitant variations' provide a basis of comparison co-ordinate with that of the presence and absence of phenomena we should expect to find not merely one method of concomitant variations, but as many methods as can be devised on the alternative basis. The analysis of Mill's Methods has brought to light two fundamental principles—the Principle of Independence and the Principle of Dependence. These, at least, may be immediately restated in terms of concomitant variation.

The Principle of Independence.—*Any circumstance which alone varies in two instances of the same unvarying phenomenon is causally irrelevant to that phenomenon.*

The Principle of Independence in this form requires instances conforming to the schema :

$$a_1b_1c_1 \;.\;.\;.\; p_1.$$
$$a_1b_1c_2 \;.\;.\;.\; p_1.$$

The subscripts, as before, indicate that we are concerned with determinate properties under the determinables, A, B, C, and P.

The Principle of Dependence.—*Any circumstance which alone varies when a given phenomenon varies is causally connected with that phenomenon.*

The schema here is :

$$a_1b_1c_1 \;.\;.\;.\; p_1.$$
$$a_1b_1c_2 \;.\;.\;.\; p_2.$$

This is the schema for Mill's 'Method of Concomitant Variations'.

If priority is to be assigned either to the procedure by cases of presence or absence or to the procedure by cases of variations there is considerable convenience in taking the latter as fundamental, and in treating the case of total absence as a limiting case in which the value of the variable is reduced to zero. That is to say, we may represent the conditions required by the Principle of Independence by

$$a_1b_1c_1 \;.\;.\;.\; p_1 \text{ or } Cv_1\text{—}p_1,$$
$$a_1b_1c_0 \;.\;.\;.\; p_1 \text{ or } Cv_0\text{—}p_1,$$

and those required by the Principle of Dependence by

$$a_1b_1c_1 \;.\;.\;.\; p_1 \text{ or } Cv_1\text{—}p_1.$$
$$a_1b_1c_0 \;.\;.\;.\; p_0 \text{ or } Cv_0\text{—}p_0.$$

In this way the special principles for presence and absence are absorbed into a more generalized account of the matter. There is also the advantage, in this formulation, that it makes explicit what is obscured in Mill's statement, viz. that the absence of a property is its absence in something to which the property is in principle applicable. Thus if the phenomenon under investigation is the fact that something is coloured, the case of absence consists in the same sort of thing becoming perfectly transparent or a perfect non-reflector of light. We could not present the shriek of a steam whistle as an instance in which the phenomenon of colour is absent.

A more radical reformulation of technical inductive methods incorporating these and other general considerations has been attempted by Johnson,[1] who replaces Mill's Methods by Four Figures of Demonstrative Induction—The Figure of Difference, The Figure of Agreement, The Figure of Composition and The Figure of Resolution. Johnson's account differs from that of Mill in three important respects. (1) The figures are based upon the principle of variation, rather than upon presence and absence. (2) They profess to be strictly demonstrative, whereas Mill leaves it unsettled whether the methods are demonstrative or problematic. (3) They depend upon a specific major premiss appropriate to the special field of inquiry rather than upon any general law such as the 'law of Universal Causation'. The specific major premiss is presumed to have been established by a prior process of induction, and the applicability of the figures is restricted to cases in which such previous inductions are at hand.

In each of the four figures the following assumptions are presupposed:

(1) That the characters under some given determinable P depend only upon characters under an assignable range of determinables, say, A, B, C, D, E.
(2) That the characters, A, B, C, D, E, are independently variable.

The first assumption must be understood to mean (*a*) that no variation in any character other than A, B, C, D, or E will effect a variation in P, and (*b*) that any variation in P depends upon at least one of the characters A, B, C, D or E. It is not assumed that variations in P depend upon each of these characters. Which characters are relevant and which irrelevant the figures are intended to reveal.

The second assumption is necessary to preserve conformity with the fundamental rule of experimental procedure, viz. that one factor only should be varied at a time. If variation in one factor involved variation in another the conclusion would be vitiated. If, for example, variations in A involved a variation in B we should be unable to assert upon which of these any consequent change in P depended. Moreover, both might involve changes in P. These changes might occur in opposed directions so that the effect of each would be cancelled by the other. In this case P would be mistakenly supposed to be independent of these conditions.

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[1] 'Logic', Pt. II., Ch. X.

For the same reason it must be further assumed in some of the figures that each factor is simple. If E, for example, were complex and resolvable to X + Y, a change in E might similarly be resolved into mutually opposed changes in X and Y.

The four figures may be conveniently considered in two pairs. The Figure of Difference and the Figure of Agreement involve the application in special circumstances of, respectively, the Principle of Dependence and the Principle of Independence. The Figures of Composition and Resolution offer technical methods for disclosing a failure to ensure single and independent variation in the relevant conditions. The former exhibits conjoint variation in two independent factors, the latter exhibits the presence of a single but complex factor.

The following schema summarizes the conditions of the Figure of Difference:

Figure of Difference.

Supreme Premiss: P depends only on A, B, C, D, E; where A, B, C, D, E are known to be constant, and D is known to be simple.

Instantial Premisses.	Immediate Conclusions.
1. A certain $abcde$ is p.	∴ 1. Every $abcde$ is p.
2. A certain $abcd'e$ is p'.	∴ 2. Every $abcd'e$ is p'.
Final Conclusion.	∴ Every $abcd''e$ is p''.

The schema exhibits two dimensions of inference corresponding to those previously distinguished as generalization and universalization. The 'immediate conclusions' are generalizations, the final conclusion is an example of universalization. Since whatever ground there may be for inferring that for some other specific value of D there will be some other value of P applies equally to all other values of D we might express the final conclusion as a complete universalization. This would read: A, B, C and E remaining constant, for every different value of D there will be correspondingly different values of P. The case differs from those discussed in § 3 in two respects. The corresponding inferences there depended upon a considerable number both of similar and varied instances, and they professed to be merely problematic. Here, on the other hand, the same conclusions are drawn from but two instances and they claim to be demonstrative. These differences will receive examination later.

As the Figure of Difference involves an elaboration of the Principle of Dependence so the Figure of Agreement is an elaboration of the Principle of Independence.

Figure of Agreement.

Supreme Premiss: P depends only on A, B, C, D, E, where B, C, D, E are known to be constant and A is known to be simple.

Instantial Premisses.	Immediate Conclusions.
1. A certain *abcde* is *p*.	∴ 1. Every *abcde* is *p*.
2. A certain *a'bcde* is *p*.	∴ 2. Every *a'bcde* is *p*.
Final Conclusion.	∴ Every *a″bcde* is *p*.

The final conclusion may again be completely universalized. This conclusion would read: B, C, D and E remaining constant, for all values of A there will be the same value of P. That is to say, the value of P is wholly independent of variations in A.

The Figures of Composition and Resolution, as we have already noted, are not figures by which generalized or universalized conclusions are established. They are figures by means of which we prove that certain conditions requisite for such inductions are absent. Both require three instantial premisses of the form:

A certain *abcde* is *p*.
A certain *abc'de* is *p'*.
A certain *abc″de* is *p*.

In this case, it will be noticed, a variation from *c* to *c'* involves a variation for *p* to *p'*, as in the Figure of Difference; but a further variation from *c* to *c″* involves no change in *p*, as in the Figure of Agreement.[1]

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[1] The identity of the premisses required by the two figures is obscured in Johnson's formulation in which the schema of the Figure of Composition fails to correspond to the descriptive text. It is, however, clear from this description that the premisses required in this figure should be the same as in the Figure of Resolution, not—as he gives them—the same as in the Figure of Difference.

Now on the assumptions underlying these methods such a trio of instances can arise only in two ways:

(1) When some other condition, say d, has varied simultaneously with c so as to result by composition in the mutual cancellation of their several effects.

(2) When the condition c is itself compound, and may be resolved into two factors, say x and y, which, again, have simultaneously varied in a manner which results in mutual cancellation. The two figures correspond to these two cases.

The Figure of Composition.

Supreme Premiss: P depends only on ABCDE where ABE are known, but D is not known, to be constant and C is known to be simple.

Instantial Premisses.

(1) A certain $abc\mathrm{D}e$ is p.
(2) A certain $abc'\mathrm{D}e$ is p'.
(3) A certain $abc''\mathrm{D}e$ is p.

Conclusion : ∴ D is an uncontrolled variable.

The symbol D in place of d is employed in this figure to indicate that it is known only that the situation involves some determinate under this determinable but that its value did not admit of precise determination. Under such conditions the 'immediate conclusions' cannot be drawn. They are in any case irrelevant under the circumstances in which the figure is required.

The Figure of Resolution.

Supreme Premiss: P depends only on ABCDE where ABDE are known to be constant but C is not known to be simple.

Instantial Premisses.

(1) A certain $abcde$ is p.
(2) A certain $abc'de$ is p'.
(3) A certain $abc''de$ is p.

Conclusion: C is compound, being resolvable into factors such as XY, where $c = xy$, $c'' = x''y''$.

The immediate generalizations are again irrelevant. They would, in fact, be invalid if C could not be defined so as to exclude the possibility of internal joint variation in its components. It should also be noted that the data do not justify any inference concerning *what* the components of C may be, nor concerning the number of components it involves.

It has been noticed that the modes of generalization and of universalization exemplified by these four figures differ from those previously examined. They differ in professing to be demonstrative on the basis of only two or three instances; whereas the forms previously considered were at most problematic even where many instances are considered.

The claims to demonstrative force in these technical figures can only be justified on the basis of some prior induction. The possibility of generalization from a pair of suitable instances may be fairly based upon the analysis of what is meant by 'dependence'. That is to say, the assertion that P depends only on ABCDE is equivalent to the alternation of universals: Either every *abcde* is p or every *abcde* is p' or every *abcde* is p'' etc. From this it follows that if any *abcde* is p then every *abcde* is p, and if any *abcde* is p' then every *abcde is* p', and so on. The only doubt is whether P in fact depends only on ABCDE. This doubt has been presumed to be removed by a prior process of induction.

So, too, with universalization. If the figures are to yield demonstrative conclusions in the second dimension of inference, it must be assumed either that the nature of dependence is such that for every difference in the determining character there is some difference in the dependent character, or that the methods are restricted in their application to circumstances in which this relation is known by prior induction to hold. In other words, it must be assumed that the prior induction guarantees that P is a non-periodic function of the determining characters. In such circumstances Johnson's 'Figures,' like Mill's 'Methods,' become strictly demonstrative. They have application when it is known that a character P is a non-periodic function of certain characters which reside within a certain assigned set. The problem is merely that of ascertaining which among this set, the determining characters are.

The conclusion to be drawn under these circumstances may be exhibited as following from a single general principle. For this purpose we may adapt the simplified and more generally useful symbolism introduced in Chapter XV. Since the number and the individuality of the characters presumed to be unvaried is irrelevant, we may express these by the single symbol, C. Let V, as before, stand for the variable determining character. The symbols, v_1, v_2, v_3, etc., will represent determinate specifications of this determining character. P, with determinates, p_1, p_2, p_3, etc., will represent the determined character and its various determinates.[1]

The general principle may then be stated as follows: *Given that a character P depends only on the conjunction of characters C and V, then it cannot be the case*

(1) *that V is simple,* and

(2) *that C is constant,* and

(3) *that there are two instances with different values of P for different values of V,* and

(4) *that there are two instances with the same value of P for different values of V.*

From this disjunctive proposition we can immediately infer four equivalent propositions in the implicative form, viz.:

I. P depending on CV, if (1), (2) and (3) are true then (4) is false.

II. P depending on CV, if (1) (2) and (4) are true then (3) is false.

III. P depending on CV, if (1), (3) and (4) are true then (2) is false.

IV. P depending on CV, if (2), (3) and (4) are true then (1) is false.

These four propositions constitute the special principles of the Four Figures of Demonstrative Induction, which may thus be restated in the following simplified forms:

I. Principle of the Figure of Difference.—*Given that a character P depends only on the conjunction of the characters, C and V, where V is known to be simple and C constant, then if there are two instances with different values of P for different values of V it may be inferred that there are no two instances with the same value of P for different values of V.*

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[1] This symbolism is specially appropriate for representing the data of an experimental inquiry. C is the group of factors intended to remain constant, V the single characters intentionally varied. In non-experimental inquiries the cases are *selected* on the basis of presumed constancy in C and presumed variation in V.

This conclusion is, of course, equivalent to the assertion that for every different value of V there are different values of P, which again is equivalent to the assertion that P is dependent on V in the sense of 'dependence' here required. This form of dependence may be expressed by saying that under conditions C, P is a non-periodic function of V.

The schema, ignoring generalizations, may assume the simple form:

$$\begin{array}{l} Cv_1 \ldots p_1. \\ Cv_2 \ldots p_2. \\ \therefore \text{ Given C, P} = f(\text{V}). \end{array}$$

II. Principle of the Figure of Agreement.—*Given that a character P depends only on the conjunction of characters C and V, where V is known to be simple and C constant, then if there are two instances with the same value of P for different values of V it may be inferred that there are no two instances with different values of P for different values of V.*

This conclusion is equivalent to the assertion that the value of P is constant for all values of V, that P is independent of V or that P is not a function of V.

This schema is

$$\begin{array}{l} Cv_1 \ldots p_1. \\ Cv_2 \ldots p_1. \\ \therefore \text{ Given C, P is independent of V.} \end{array}$$

III. Principle of the Figure of Composition.—*Given that a character P depends only on the characters C and V, where V is known to be simple (but C is not known to be constant), then whenever there are two instances with different values of P for different values of V and two instances with the same values of P for different values of V it may be inferred that such instances involve a variation in C.*

The simplest case in which there are two instances with different values of P for the different values of V and two with the same values of P for different values of V is that in which there are three instances involving three values of V and two of P, as in the following schema:

$$\begin{array}{ll} Cv_1 \ldots p_1 & \\ Cv_2 \ldots p_2 & \text{(Where V is known to be simple.)} \\ Cv_3 \ldots p_1 & \\ \therefore \text{ C is not constant.} & \end{array}$$

This formula is somewhat more comprehensive than that adopted by Johnson. In his account the special case is selected in which C is analysed into simple components all except one of which is known to be constant. The conclusion drawn is then that this remaining component as well as V has varied. This, however, is a rare and artificial case. The chief use of the figure is to exhibit accidental variations. The typical case is that of experimental inquiry in which V is a property varied by intention and C a group of properties intended to be constant. The figure thus exhibits a failure in respect of this intention. In such cases it is not usually known in what the accidental variation more precisely consists. All that we know is that it lies somewhere in the set of conditions C. Its more precise determination is a matter for further investigation.

IV. Principle of the Figure of Resolution.—*Given that a character P depends only on the conjunction of characters C and V, where C is known to be constant (but V not known to be simple), then whenever there are two instances with different values of P for different values of V and two instances with the same value of P for different values of V it may be inferred that V is complex and that more than one of its components have simultaneously varied.*

The schema is :

$Cv_1 \ldots p_1$
$Cv_2 \ldots p_2$ (Where C is known to be constant.)
$Cv_3 \ldots p_1$
$\therefore$ V is complex.

It is evident that in many circumstances it is not possible to be certain that C is constant or that V is simple. In which case the conclusion to be drawn from three instances of the type required by the figures of composition and resolution is the alternative proposition : Either C is not constant or V is not simple.

§ 6. Determinate Functional Laws.

In Mill's ' Methods ' and Johnson's ' Figures ' we have two examples of the attempt to escape the besetting difficulties of inductive inference, viz. dependence upon a complex body of evidence, and the merely problematic conclusion. In the Methods and the Figures two or three instances only are required and the conclusion claims to be demonstrative. But such devices, it would seem, obtain their advantage only at the cost of practical utility. It may be doubted, in particular, whether a given method or figure is applicable. And it is doubtful, in general, whether the underlying assumptions may be made.

The merits of such formulations are that they render explicit the general requirements of the case, that they enable us to distinguish the demonstrative and the problematic phases of a total inductive inquiry, and that they define a certain limit to which in certain special cases we may approximate. Johnson's Figures, in particular, may be defended on the ground that they embody important refinements on the methods suggested by Mill, and that they define the requirements of proof in the case of a special type of function.

They are, however, even under the most favourable conditions, inapplicable to periodic functions,[1] and they yield at best the indeterminate conclusion that P is *some or other* function of V.

To determine periodic functions or the specific form of any function whatever a more elaborate process is required. This involves precision in measurement, a larger selection of instances, and the abandoning of all claims to demonstration. The procedure involves an extension of the Figure of Difference, instances being collected which conform to the following schema :

$$\begin{gathered} Cv_1 \text{ is } p_1 \\ Cv_2 \text{ is } p_2 \\ \cdots \\ Cv_n \text{ is } p_n \end{gathered}$$

in which the subscripts stand for precise measures of the values of V and P.

The problem is then to ascertain a determinate formula or equation which fits the given cases. The solution depends, in part, upon the number of values involved and the precision with which they have been ascertained. But however many measures have been taken, they must at least be finite ; and however careful the measurements they can never be quite precise. In consequence, it will always be the case that there are many different mathematical functions consistent with the data. The question then arises : Which formula is ' right ' ?

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[1] Except on the assumption that periodic functions can result only from complex variables.

Two criteria are commonly suggested—the criterion of simplicity, and that of analogy. The scientist will accept whatever formula is the simplest, and that which is suggested by known analogous laws. Neither criterion, however, will yield demonstrative conclusions. In fact, if we mean by the ' right ' formula that which is certainly true, the right formula is one which from the nature of the case never can be known. To assert that many different formulæ are consistent with the known facts is equivalent to asserting that no one formula is deducible from these facts.

The simplest formula is not necessarily correct when applied beyond the cases from which it is derived. Simplicity, moreover, is relative to the data. What is the simplest formula for n values of the variable may not be the simplest applicable to $n + 1$. The importance of simplicity in a formula arises solely from the fact that a formula has a *use*. It is employed for deducing other values of the variable and for inferring what would happen under certain conditions. Simplicity in a formula, therefore, economizes labour.

The use of analogy consists in reference to other formulæ known to apply in ' similar ' cases. It must therefore be subject to all the general conditions of argument from analogy, and in particular to the conditions which distinguish an ' important ' from a merely superficial analogy. In seeking for a formula which expresses the facts concerning the radiation of heat we might attend either to the analogies provided by the emission of light from a source, or to those provided by the emission of sound or smell. It is clear that superficial analogies are of less value than those which depend upon the discovery of some fundamental identity in the kinds of facts considered.

Moreover, the appeal to similar cases cannot provide a complete solution of the problem. The difficulty is one which affects all formulæ alike, including those to which the argument from analogy appeals. The conclusion remains ineradicably problematic, though at this level of inductive inquiry the probabilities attained are very much higher than those which may be reached by the procedure of direct induction.

The forms of induction dealt with in the earlier sections of this chapter yielded conclusions which were both problematic and indeterminate. Those of paragraph 5 were demonstrative but remained indeterminate. Those with which we have been concerned in the present paragraph are determinate but only problematic. Beyond this, it would seem, inductive procedure cannot go. An inductive conclusion which is both determinate and demonstrative is a limit to which we can only approximate. In accordance with the principles noted in the first section of this chapter, we may increase the measure of probability by accepting a less determinate formula; and we can increase the determinateness at a sacrifice of probability. We can also simultaneously increase the probability and the determinateness of our conclusion by increasing the number of our instances and the precision of our measurements. It is in this sense chiefly that *number* is important; but, even so, what counts is the number of different values which our variables assume, not the number of precisely similar cases we examine.

BIBLIOGRAPHICAL NOTES.

CHAPTER XVI.

This chapter is intended to provide an introduction to the relevant logical theory pertaining to the transition from the elementary generalization to the form *All S is P* to the genuine functional law—a transition of fundamental importance to the progress of any science.

Of this subject only the fringe is touched by Mill's treatment of the Method of Concomitant Variations. In this field **Jevons, Principles of Science** (especially Book IV.), is far superior to Mill. Mill, it is always to be remembered, was concerned with the study of induction less for its own sake or the sake of its bearing upon the foundations of the natural sciences than for the contribution it might make to the progress of the social sciences. Jevons, on the other hand, had much the more intimate acquaintance with the more exact empirical sciences, and a clearer apprehension of the importance of the functional law.

The determination of relations of functional dependence may proceed either along lines on which the inference approximates to the limiting case of demonstrative induction or it may be effected in accordance with the methods of statistical reasoning. On the former case the outstanding authority is **Johnson (Logic,** Part II., Chapters X. and XI.), with whose treatment the present chapter is in the main concerned. It should be noted that although the conception of universalization as developed in this chapter is derived principally from Johnson the term itself is not. Johnson, in fact, employs this term for the process of generalization constituting the 'first dimension of inference' in the total process here considered. Cf. **Logic,** Part II., Chapter V., page 216. The connection between universalization and the proof of Causal Laws is discussed by the present writer in the **Proceedings of the Aristotelian Society,** Vol. XXXII. **(Hume's Doctrine of Causality).**

A discussion of the proofs of functional dependence by statistical methods would require a detailed exposition of the theory of correlation (cf. *supra*, Chapter X.). This aspect of the question is more prominent in the chapter on Causal Determination and Functional Analysis in **Stebbing's Modern Introduction to Logic**, Chapter XVIII.

On the special subject of Demonstrative Induction and its functional extension the more advanced student should not fail to consult **Dr. C. D. Broad's** important articles in **Mind** (Vol. XXXIX., Nos. 155-156)

CHAPTER XVII.

ANALOGY.

§ 1. General Characteristics of the Argument from Analogy.

THE most familiar form assumed by the argument from analogy is that which consists in inferring that, since one thing has a certain property, therefore a certain other thing also has that property. *The earth is inhabited: therefore Mars is. X is endowed with a mind and a soul: therefore X's dog is so endowed.* These are inductive arguments in the sense previously defined. They involve some measure of generalization. At first sight, they appear to differ in several respects from the ordinary cases of induction. In other cases there is an explicit reference to a connection between two properties. Whatever has the property p is asserted to have the property q. In analogy, on the other hand, the immediate reference is to two *particulars* and, at first sight, to only one property. Again, in the common forms of inductive argument the conclusion is based upon a multiplicity of instances, and generally upon a premiss which asserts that *every* examined instance of P is an instance of Q. Here, on the other hand, the whole weight of the proof seems to rest upon a single instance. Similarly with the conclusion. In the one case, the conclusion embraces *every* instance of a certain sort; in the other, the transition is restricted to a single novel instance.

These differences, however, are superficial; and they disappear upon analysis. They arise merely from differences in the purpose governing the inferential process. There is no fundamental difference in the logical character of the operations involved.

An argument from analogy, like an ordinary generalization, relates to a conjunction of at least two properties. To symbolize the former by S_1 is p ∴ S_2 is p obscures a feature of fundamental importance. It obscures the fact that S_1 and S_2 are known to possess at least one character in common other than the character p. We cannot infer indifferently that, since the Earth is inhabited, therefore Mars, Saturn, the Sun or the Moon is also inhabited. The argument that Mars is inhabited is plausible only in virtue of some better resemblance between the Earth and Mars than obtains between the Earth and other heavenly bodies. The argument must therefore be symbolized

S_1, which is A, is P.
∴ S_2, which is also A, is probably also P.

In common, therefore, with the ordinary generalization there is a reference to some connection between two distinct properties.

Consider next the second contrast, viz. that between the number of instances occurring in the premisses. The instantial premiss, *Every examined A is P*, does not specify any particular number of cases. If an unspecified number of instances can confer probability on the conclusion, it is difficult to deny that a single instance confers at least some probability as well. Conversely, if the argument from analogy presupposes that a single instance confers probability, it is reasonable to suppose that a larger number of instances may increase it. If, for example, we argue from analogy that Mars has an elliptical orbit because the Earth has, the inference is presumably strengthened by showing that several other planets have an orbit of this kind. The conclusion is still further strengthened by showing that every other planet has an elliptical orbit. An extension of the argument from analogy by examining every available instance assimilates it to the type of inductive argument which may be expressed in the form: *Every examined A is P*, therefore, *the next A will be P*.

Similar considerations apply to the third difference between the two cases. In the typical inductive argument the conclusion is that *every A is P;* in the typical argument from analogy the conclusion is *that a certain unexamined A is P*. But it is evident that this conclusion, if valid at all, is valid irrespective of which particular A may happen to be selected. It would apply to any similar A. We may therefore include under arguments from analogy inference of the form: *A certain A is P* ∴ *Every A probably is P*.

If any distinction is to be drawn between the ordinary inductive argument and the argument from analogy, it may most conveniently be based upon the *number* of characters involved. In general, selecting as the limiting case the inference *A certain examined A is P* ∴ *A certain other A is P*, it would seem that we may strengthen the argument in two ways. We may either increase the *number of instances* upon which the conclusion is based, or we may increase the *number of characters* known to be possessed in common. In so far as we adopt the first procedure we should be said to be arguing by 'pure' induction. In so far as we adopt the second procedure we are arguing from analogy. It is important to notice, however, that in both cases at least two instances are involved, and at least two characters. The relation between the two types of inference is well stated by Johnson: 'But it is essential to insist that neither by accumulating instances alone, nor by accumulating adjectives alone, can any inference be drawn, and that inference of this type, by whatever name it might be called, is governed by principles which underlie both induction and analogy—requiring an intensional as well as an extensional link. For example, no mere accumulation of instances $s_1, s_2, s_3 \ldots s_n$ that are p could give any probability that a new instance s will be p, unless s were known to have at least one character predicable of all these instances. And conversely, no accumulation of characters $p_1, p_2, \ldots p_m$ that are predicable of s could give any probability that a new character p is predicable of s, unless p were known to be predicable of at least one instance having all these characters'.[1]

It will be convenient to extend further the conception of an argument from analogy so as to take explicit account of its negative form. The more familiar case can be expressed in the form:

S_2 resembles S_1 in respect of the characters *abc* . . .
∴ S_2 resembles S_1 in respect of the characters *pqr* . . .

Since any point of resemblance constitutes a basis for an argument in favour of further resemblances, any point of difference will constitute a basis for expecting further differences. We obtain then the form:

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[1] Johnson, 'Logic', III., pp. 46-47. (Johnson speaks of properties or characters as 'adjectives'.)

S_2 differs from S_1 in respect of abc.
$\therefore$ *S_2 differs from S_1 in respect of pqr.*

Since, further, any two objects (or sets of objects) we choose to compare will be found to agree in certain respects and to differ in others, the same instances will provide the basis both for a positive and a negative argument from analogy. The case will then assume the following form: *Since S_2 resembles S_1 in respect of abc there is some presumption that it agrees in respect of pqr. Since S_2 differs from S_1 in respect of kln there is some presumption that it differs in respect of pqr.*

A resemblance or a difference may arise in two ways. S_2 may resemble S_1, (*a*) in possessing some characters which S_1 possesses, (*b*) in lacking some characters which S_1 also lacks. Similarly, differences may arise either (*a*) through S_2 lacking some characters which S_1 possesses, or (*b*) through S_2 possessing some characters which S_1 lacks. Consequently, in assessing the relative weights of the positive and the negative arguments from analogy there are four elementary types of premiss to be considered:

1. S_2 is characterized by *a* which also characterizes S_1.
2. S_2 is not characterized by *b* which also does not characterize S_1.
3. S_2 is not characterized by *c* which characterizes S_1.
4. S_2 is characterized by *d* which does not characterize S_1.

The former two of these strengthen the positive argument from analogy, the latter two strengthen the negative argument.

Where, as is often the case, the instance under consideration may be compared with two other instances one of which possesses the character in question and the other of which lacks it, the four types may be combined in support of the positive argument. Thus, let the question be whether S_2 possesses the character *p*. Two instances, S_1 and S_3, are available for comparison. S_1 is known to possess the character *p*; S_3 is known to lack it. The four following premisses then support the positive argument, viz. that S_2 does possess the character *p*.

(i) S_2 is characterized by *a* which also characterizes S_1.

(ii) S_2 is not characterized by *b* which also does not characterize S_1.

(iii) S_2 is characterized by *c* which does not characterize S_3.

(iv) S_2 is not characterized by *d* which does characterize S_3.

Mutatis mutandis, corresponding premisses can be constructed in support of the negative argument.

The procedure which consists in assessing the relative weights of the positive and negative arguments from analogy contributes in some measure to a solution of a problem raised in Chapter IX. and again in Chapter XIII—a problem to which as yet we have found no solution. It was there noted that in problematic inference one of three conditions should be satisfied. Either we should be able to assign a numerical value to the probability of our conclusion, or we should be able to assert that it is more or less probable than some accepted standard, or we should be able to say that the conclusion was more or less likely to be true than false. The double form of the argument from analogy indicates one condition at least in which the third of these alternatives may be realized. When every resemblance favours equally the conclusion that a given thing has a certain property and every difference weakens the presumption by an equal amount, the final conclusion to be drawn may be based simply upon the relative weights of the positive and the negative arguments. We may be able to assert that the presumption that the object has the property in question is greater than the presumption that it lacks it. This is the kind of conclusion which in practical affairs is commonly required.[1]

§ 2. Principles of Reasoning from Analogy.

The degree of probability attaching to an argument from analogy depends partly upon the nature of the conclusion and partly upon the nature of the premisses by which it is supported. In discussing these conditions we need consider only the positive argument from analogy. In general what applies to the positive argument applies, *mutatis mutandis*, to the negative argument. Circumstances which strengthen the positive argument weaken the negative, and *vice versa*. It will be convenient also to restrict attention to the case in which a general conclusion is drawn on the basis of a number of observed instances. The argument is then of the form :

Certain examined instances of *abc* are *pqr*.
∴ Everything which is *abc* is *pqr*.

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[1] In practice, however, it is rarely possible to assume that all resemblances and all differences are of equal weight. Cf. § 3.

It will be found that the probability attaching to a conclusion of this kind depends in part upon the *comprehensiveness* of its terms, irrespective of the special character of the premisses adduced in its support. A term is more comprehensive according as it signifies a larger conjunction of characters, and it is less comprehensive in so far as it signifies a smaller conjunction of characters. The conjunction *abc* is more comprehensive than the conjunction *ab*, which in turn is more comprehensive than *a*.

It is easy to see that more evidence is required to support a generalization in which the predicate is more comprehensive than to support one in which the predicate is less comprehensive. It would be more difficult to prove that Mars is inhabited by beings like man than to prove that it is inhabited by some form of living thing. It would be more difficult to establish the fact that a picture was painted by Rembrandt than to prove that it was painted by some one or other belonging to the Dutch school.[1] In general it is more difficult to prove that every *abc* is *pqr* than to prove that every *abc* is *p*.

On the other hand, it is easier to establish a generalization with a more comprehensive subject than it is to establish one with less comprehensive subject. It requires less evidence to prove that all vertebrates resemble man in possessing minds than to prove that all animals possess minds. It is easier, in general, to prove a conclusion of the form *every abc is p* than one of the form *every a is p*. The latter is a wider generalization. We find, therefore, that the more comprehensive the subject term of the generalization and the more comprehensive its predicate term the greater the probability of this generalization.

We may consider next how the probability of the conclusion is affected by the comprehensiveness of the terms of its premiss. Here it will be found that the *more* comprehensive the predicate and the *less* comprehensive its subject the greater the evidential value of the premiss. If we find that in all examined cases certain features in a painting are not merely characteristics of Rembrandt but are characteristic of Rembrandt at a certain period of his life, this is very favourable to the less comprehensive conclusion that a picture exhibiting these characters was painted by Rembrandt at some time or other. Similarly, if the marks are such as point to Rembrandt's work we may have more confidence in the less comprehensive conclusion that the picture was painted by some one of the Dutch School. Conversely, the less comprehensive the subject of the premiss the more favourable is the evidence for the conclusion. If we have good reason to suppose that mind is present in a great variety of forms of animal life, i.e. forms having very little in common, this is better evidence for the conclusion that mind is present in a new case than if the evidence was drawn from a smaller group of instances, i.e. from instances having much more in common.

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[1] Notice that, whereas the *conjunction abc* is more comprehensive than *a*, an *alternation*, *a* or *b* or *c*, is less comprehensive than *a*. It should also be noticed that, throughout the present context, it is assumed that the characters conjoined are independent. It is not more difficult to prove something concerning three-sided triangles than it is to prove something concerning triangles, since the additional property, three-sidedness, is not independent.

We thus obtain two general principles with regard to the dependence of the probability of the conclusion upon the comprehensiveness of its terms and of the terms of its premiss.

I. *The more comprehensive the subject of the conclusion and the less comprehensive its predicate the greater the probability of that conclusion.*

II. *The less comprehensive the subject of the premiss and the more comprehensive the predicate of the premiss the greater the probability of the conclusion.*[1]

From these general principles it may be seen that taking any body of evidence as a fixed datum we can obtain a series of conclusions of varying degrees of probability according to the comprehensiveness of their terms. The more comprehensive we are willing to make the subject, and the less comprehensive we are willing to make the predicate, the greater will be the probability—up to a certain limit—of the conclusion we obtain.

But in most practical and scientific inquiries we are not restricted by a fixed body of evidence. Commonly, the inquiry takes the form of the question: Is such and such a thing characterized by the property p, or is it not-p? Or we ask the question: What sort of thing has the property p? In the case of both questions we are prepared to modify our account of the subject. When we are concerned with a single given thing, we are prepared to consider its other properties and so reach the conclusion that it is probably p because it is *abc*, or because it is *abcde* . . . etc. When we ask: What sort of a thing is p? we are prepared to accept various answers such as that *everything that is a is p* or *everything that is abc . . . is p*, etc. The premisses are fixed least of all. As a basis for our conclusion we are prepared to search for new instances of things that are p.

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[1] These principles should be compared with those relating to the *determinateness of terms*, p. 315-16, *supra*. The similarity and the difference between the two cases can be expressed by saying that an indeterminate characteristic is equivalent to an alternation of characters under the same determinable, whereas a determinate characteristic restricts the number of possible alternants, in a manner analogous to that in which a less comprehensive characteristic restricts the number of possible characters under different determinables.

We may, therefore, inquire more precisely into the procedure whereby, given *prima facie* evidence that a certain thing, S_n, is p because it is a, we attempt to strengthen the argument that this thing, or all things of a certain sort, are p. Under these circumstances the argument can be strengthened in two ways:

(1) By finding fresh points of resemblance between S_n and other things that are a.
(2) By finding fresh instances of things that are p.

Our point of departure may be represented as comprising the following data:

S_1, S_2 . . . S_k agree in *abcdef* and p.
S_n is a.

In the first case we strengthen the argument that S_n is also p by finding other points of resemblance to S_1, S_2, . . . S_k. We observe, for example, that S_n is also b, c and d. In this case we may be said to have increased the *positive analogy* between the given case and the evidential instances. This, it will be noticed, is the same thing as increasing the comprehensiveness of the subject of the conclusion.

In the second case we strengthen the argument by finding other instances of things that are p, e.g. S_l, S_m, . . . etc. This is important precisely in so far as it increases the variety of the instances, i.e. in so far as it decreases the number of characteristics common to the instances, or in so far as it increases the number of respects in which the evidential instances differ among themselves. In this case we may be said to have increased the *negative analogy* between the evidential instances themselves.[1] This is equivalent to decreasing the comprehensiveness of the subject of the premiss.

We thus strengthen the positive argument from analogy by increasing the positive analogy between the instances of the premiss and the instances of the conclusion, or by increasing the negative analogy between the evidential instances themselves.

In combining these two operations the comprehensiveness of the subject of the premiss increases and the comprehensiveness of the subject of the conclusion decreases. In other words, they converge to the point at which they become identical. It is instructive to consider this special case. It constitutes what Keynes[2] describes as a *perfect analogy*.

A perfect analogy could be obtained only if we had complete knowledge of the characters of the instances. If there was any character concerning which we did not know whether the instances possessed it or not we should not know whether it belonged to the positive analogy or to the negative analogy between these instances. More precisely a perfect analogy requires three conditions to be satisfied:

(1) That we should possess complete knowledge of the characters possessed by each of the instances.
(2) That the generalization should be co-extensive with these common properties, i.e. that every common property should be either part of the subject of the generalization or part of its predicate.
(3) That all the instances should be favourable to the generalizations.

Under such circumstances the argument from analogy attains its maximum weight and it cannot be strengthened by the addition of further instances. The general thesis of Mr. Keynes is that all arguments from analogy depend for their power upon the extent to which they approximate to the conditions of a perfect analogy. All ordinary arguments suffer from limitations which arise from the fact that our knowledge of the instances is limited. Consequently, we can never be sure that the instances agree only in respect of the characters concerning which we wish to generalize. We increase the number of instances because every such increase improves the chance that the negative analogy between the instances will be thereby increased. In other words, the more instances we consider the greater the chance that the comprehensiveness of the subject of the premiss will be reduced to the point at which it is identical with that of the conclusion.

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[1] In speaking of the positive or negative analogy it is important to make clear whether we are referring to the analogy between the evidential instances and the instances of the conclusion, or the analogy between the instances alone.

[2] 'Treatise on Probability', Pt. III., Ch. XIX.

It is further pointed out by Mr. Keynes, however, that if we did achieve the requisite logical conditions of the perfect analogy it would then lose its practical utility. If we include in the subject of our generalization *all* the characters common to the set of evidential instances we shall find that anything to which this generalization applies must be one of these instances themselves. For example, if we attempted to apply the generalization to something in a different place and time from any of the instances considered this thing would differ in some respect from all the instances. They all agreed at least in the negative property of not being in that place at that time. Consequently, in a perfect analogy this character must be included in the generalization.

It is argued, therefore, by Mr. Keynes that the use of the argument from analogy requires the assumption that some of the common features of the evidential instances may be regarded as irrelevant. In consequence, new cases to which the generalization may be applied may differ from all the examined instances in respect of these irrelevant characters without loss of probability in the conclusion.

§ 3. Analogy and Direct Induction.

It has been seen that a single instance of something which is both *a* and *p* yields some slight presumption in favour of the generalization: *Every a is p*. We may endeavour to increase this presumption either by examining other instances of *a*, or by reviewing other properties of the things that are *a*. In the first case we propose to support the generalization by showing that in every one of a large number of examined instances *a* is accompanied by *p*; in the second case we hope to show that the instances adduced as evidence agree in so few respects that whatever property is absent *p* is always present. The first procedure relies upon simple enumeration as a logical principle, the second relies upon the principle of analogy. We may proceed next to inquire which of these two principles provides the more fundamental basis for generalization.

As has been already noticed, the two principles are not wholly independent. In selecting instances in accordance with the principle of simple enumeration we can do so only on the basis of observed similarity of *character*; and in the procedure of analogy we are led to seek *new instances* which lack some property common to those already known. Nevertheless, it may be said that the argument from analogy is the more fundamental, since an increase in the number of instances examined increases the probability of the conclusion only in so far as it augments the negative analogy. No value attaches to a new instance which merely duplicates one that is already known. A single additional instance which reduces the comprehensiveness of the subject of the premiss in respect of two characters is of greater value than any larger number of instances which jointly reduce it in respect of only one.

It is for this reason that analysis is of greater importance than mere enumeration. It is better to enumerate a smaller number of instances the characters of which are as nearly as possible exhaustively described, than to enumerate a large number of instances of which only a few characters are known.

But not every increase in the comprehensiveness of the subject of the conclusion nor every decrease in the comprehensiveness of the subject of the premisses is of evidential value. Such changes are important only when they relate to *independent* characters. It is obvious, for example, that in any argument from the analogy between the Earth and Mars, we can increase the enumeration of similarities by mentioning characters implied by those already known. Both bodies are spherical. In consequence they have in common all the properties of a sphere. Of these we could enumerate an indefinite number. To do so, however, would not strengthen the argument. The only important way of increasing the comprehensiveness of the subject of the conclusion is to enumerate further independent properties which the two objects share. Conversely, the only important way of reducing the comprehensiveness of the subject of the premiss is to add further independent properties to the negative analogy between the examined instances.

These considerations may be summed up in two general principles relating to the evidential value of new instances and new characters :

(1) An additional instance augments the probability of the conclusion when it lacks at least one character common to all instances previously examined.

(2) An additional character (introduced into the positive analogy between the subjects of the premiss and conclusion, or introduced into the negative analogy in the evidential instances) augments the probability of the conclusion when it is independent of the characters previously enumerated.

The principles of analogy are relevant also to an inductive procedure which has been previously described, but which we have left in a state of incomplete analysis. It has been observed that we are sometimes in a position to infer that certain unexamined members of a given class possess a certain character even when we know that the character in question does not belong to every member of that class. We infer that certain children of criminal parents are likely to manifest criminal tendencies in spite of many exceptions to the general rule. Similarly, we may infer that a given newly discovered element will probably have an atomic weight expressible in a whole number by a scale on which oxygen is 16. This inference is based on the knowledge that many elements conform to this rule though there are important exceptions.

Inference of this kind suggests that there may be a valid process of inference from ' particulars to particulars ', independently of the possibility of establishing any general rule. But this interpretation would be mistaken. It has already been noticed that whenever it is permissible to infer that a given thing has a certain property the principle which justifies the inference always warrants a generalization to the effect that *everything of a certain sort has that property*. Thus if it be permissible to infer, on the basis of examined instances, that a given element has an atomic weight expressible by a whole number on a certain scale, it is permissible to infer that all things of a certain kind have this property. The difficulty here is to specify the kind.

Clearly it cannot be that all elements have this property. This generalization is too wide, and is disproved by the exceptions. We can, of course, infer that all elements exactly like those known to possess it will also have the property. But this generalization is too narrow. It has been seen that generalizations of the form : *Everything which is exactly like the members of a given set has a certain property* must always be trivial. A generalization of this kind applies to nothing outside the set in question. Anything which possesses all the properties common to a given set must be a member of that set. Any element exactly like the members of an examined set of elements will be a member of that set. We have, therefore, to find a generalization intermediate between that which is too wide and that which is too narrow.

Stated as a problem in analogical reasoning the data are as follows :

(1) A set of things (elements), agreeing only in the characters *a*, *b*, *c* (characters common to all known elements), do *not* agree in possessing the character p (having atomic weight expressible as a whole number).

(2) A sub-set of these things, agreeing in the characters *a*, *b*, *c*, *d*, *e*, *f* (a more comprehensive group of characters), do agree in possessing p.

We propose to infer that another thing, S_r (a newly discovered element), probably possesses the character p. The problem is, therefore, to select a legitimate generalization which will cover the case of S_r. The generalization *Every abc is p* is too wide, since it includes cases which are not p. The generalization *Every abcdef is p* is too narrow, since it would exclude S_r. The first step is to examine the characters of S_r. If we find that it agrees with the subset only in *abc* this would preclude the inference that S_r is p, since we have already reason to suppose that the corresponding generalization is false. If, on the other hand, we find that it agrees in, say, *abcde*, we can infer that *every abcde is p* with the same assurance that we can infer that S_r is p. In other words, we infer that the next element probably has an atomic weight expressible by a whole number only on the supposition that there is some definable sub-class of elements of which this is true. This class is definable, perhaps, in terms of certain characteristic features of sub-atomic structure.

Wherever inference appears to proceed from 'particulars to particulars' this possibility of generalization is present.

The difficulty in effecting it is derived from the fact that the class of cases with which it is concerned requires to be simultaneously defined. Commonly it will not correspond to any group of things previously distinguished by a familiar class term. The inductive operation has two phases. It defines a novel class and it effects a generalization with regard to this class.

The same general considerations may apply to statistical inferences. Whenever we find, for example, that 50 *per cent. A's are p's* we may expect to find some sub-class, say ABC, such that the generalization may be made: *every ABC is P*. Commonly, this is effected in several stages: 50 *per cent. A's are P*; 70 *per cent. AB's are P*; 100 *per cent. ABC's are P*. Theoretically, every statistical generalization may be so resolved into some complete induction. The final generalization may, of course, be of an extremely complicated kind.

§ 4. Analogy and Relations of Dependence.

In the *pure* argument from analogy—an argument which rests on the principles of analogy alone—every resemblance and every difference between the instances contributes either to the positive or the negative argument, and contributes to the same extent. Whereas pure induction depends on the number of *instances*, pure analogy depends on the number of *characters* considered.

It is obvious, however, that in practice few arguments conform to this simple description. It must at least be assumed that certain resemblances between the evidential instances, and certain differences between these and the new instances included in the generalization, may be disregarded as irrelevant. Such irrelevance cannot be established by an argument from analogy, It must be independently known.

A further obvious feature of all ordinary arguments from analogy is that more weight is attached to certain similarities or differences than to others. In an argument to show that Mars is inhabited because the Earth is inhabited, more importance is attached to characters known to be connected with life than to those in which no such connection is seen. It would be more important, for example, to show that Mars resembles the Earth in possessing atmosphere and a certain moderate temperature than to show that it resembles the Earth in shape and size.

Important properties are thus properties which have some mode of connection with the inferred property. This mode of connection may be defined in terms of implication or dependence.

With regard to a given property, A, and an inferred property, P, there are three relations to be considered:

(1) that A and P are independent,
(2) that A implies P,
(3) that A is implied by P.

Mr. Keynes postulates that in an argument from analogy we may assume that position in time and space may always be disregarded as irrelevant. This means that, if P be any character whatever and A the character of being in a certain place or of existing at a certain time, A can always be eliminated from any statement of analogy. If A be any part of the positive analogy between the evidential instances and the instances comprised within the conclusion, or if it be any part of the negative analogy between the evidential instances themselves, it contributes nothing to the argument. Conversely, if A be any part of the negative analogy between the evidential instances and the instances of the conclusion, or if it be any part of the positive analogy between the evidential instances without entering into the terms of the generalization, it does nothing to weaken the argument. This, if true, is very important. It means that we can infer from certain instances to certain other instances disregarding the fact that the two sets of instances differ in spatial or temporal position. It invalidates objections to an inductive argument based on the fact that the evidence is drawn from the past whereas the conclusion embraces the future, or on the fact that the evidence is drawn from instances in some parts of space while the conclusion covers instances in different parts of space. The basis of this postulate will be discussed later. Clearly it is not guaranteed by the principles of analogy so far defined.

If we are permitted supplementary knowledge concerning the independence of properties it is plausible to suppose that we sometimes have supplementary knowledge concerning their relations of dependence. Suppose it is known that A implies P, i.e. that if A is present P is present also. This, too, would be important; so important, in fact, that it would enable us to dispense with the argument from analogy altogether. If we know that A implies P, and know that A is present we can infer that P is present without regard to other properties of the instances examined. Thus if we know that the presence of an artificial construction, such as a canal, implies the presence of life, and we know that Mars possesses canals, there is no need to pursue the argument from analogy. We may directly infer that Mars is inhabited.

Corresponding considerations apply in the case of knowledge that A is implied by P. Here, it is the negative argument from analogy that is put out of court as wholly superfluous. If we know that P implies A the absence of A alone implies the absence of P. Given that the presence of life presupposes a certain appropriate temperature, the discovery that a given planet lacks that temperature proves that planet to be uninhabited, however close the analogy with the Earth may otherwise be.

If, on the other hand, we know that P implies A, and that A is *present* how does this affect the argument?

Clearly, of two properties one of which, A, is known to be implied by P, and the other of which, M, is not known to be implied by P, the presence of the former strengthens the argument from analogy to a greater extent than the latter does. It does so because the problem is, in effect, to establish a proposition of the form: If ABC . . . then P; i.e. to define a set of conditions which jointly are sufficient to determine P. If we have antecedent knowledge that A is one of these conditions our problem is correspondingly reduced.

It is thus evident that supplementary knowledge of dependence and independence is highly relevant to an argument from analogy. The important point, however, is that such knowledge *is supplementary*. The argument from analogy yields demonstrative proof that P is independent of any character which is part of the negative analogy between the evidential instances themselves. It does not yield demonstrative proof that it is independent of anything comprised within the positive analogy. That P is implied by some character or group of characters is precisely what the argument is intended to establish. To incorporate such knowledge in the principle of analogy is in effect to assert that in order to show that some collection of characters determine some other property we must already know that some of these characters determine it in part.

If antecedent knowledge of this kind must be granted before the argument from analogy can begin we must agree with Mill that the principle of analogy cannot provide the ultimate foundation of induction.

Mill was no doubt mistaken in supposing that analogy was merely an inferior substitute for the procedure of induction which he himself expounded. The fact is rather that both indirect induction and the argument from analogy depend upon some more ultimate principles. They are both, in fact, on a similar footing. This is brought out by the consideration that the positive argument from analogy is an extension and a refinement of the Method of Agreement, or more strictly a special application of the Principle of Independence.

In the positive argument from analogy we ignore cases in which the character P is known to be absent. We employ the principle of independence with a view to establishing dependence. The cases selected conform to the schema :

$$Cv \ldots p.$$
$$C\bar{v} \ldots p.$$

So far we have been concerned only with the conclusion that p is independent of v. In terms of the principles of analogy V is shown to be part of the negative analogy between the two instances adduced as evidence. But given that p is determined by CV, it follows that P is implied by something within the collocation C. We proceed to analyse C into its constituents, say, *abcdef*. This collocation constitutes the positive analogy between the instances selected. We may now continue to re-apply the principle of independence. We may vary f and find that P remains unchanged.

$$(abcde)f \ldots p.$$
$$(abcde)\bar{f} \ldots p.$$

These instances show that f also is part of the negative analogy.

Next we vary e, and then d, thereby progressively increasing the negative analogy. We may approximate to the point at which the factors which remain are identical with those which are comprised within the scope of the proposed generalization. We may conclude, for example, that everything that is *abc* is p.

The conclusion is only problematic because the approximation is always incomplete. Complete correspondence is prevented by the considerations which prevent the attainment of the condition of a perfect analogy, and because the analysis of the instances is always incomplete. Moreover, the procedure ultimately depends on the assumption that P is, in fact, determined, i.e. upon the assumption that the positive analogy cannot be reduced to zero. Into the ground of this assumption we shall inquire in the following chapter.

BIBLIOGRAPHICAL NOTES.

CHAPTER XVII.

The principles of reasoning by analogy have received some recognition from most writers on Induction, but in too many cases Analogy has been regarded as only a subsidiary or derivative form of induction. Of this tendency Mill's treatment (**Logic**, Book III., Chapter XX.) is representative. A more adequate discussion is provided by **Jevons** (**Principles of Science**, Book V.). The assertion of Jevons that ' In generalization the resemblances have great extension and usually little intension, whereas in analogy we rely upon the great intension ' anticipates the general position of **Johnson** (**Logic**, Part III., Chapter IV.), in whose exposition direct generalization and analogy are presented as co-ordinate and interdependent modes of inference

It has remained for **Mr. J. M. Keynes** to provide in his monumental **Treatise on Probability** a detailed treatment of the subject in relation to the general theory of probability. This work contains an almost exhaustive bibliography ; but the comparative neglect by logicians of the problems of analogical reasoning is exhibited by the limited number of items in this bibliography which pertains to this subject.

CHAPTER XVIII.

LOGIC AND THE STRUCTURE OF THE UNIVERSE.

§ 1. The Problem of Induction: Logical Conditions.

INDUCTIVE arguments, ignoring universalizations, may be broadly classified according as they depend upon one of three types of premiss or issue in one of three types of conclusion. The premiss may be of the form: *Every examined S is P*, *A certain percentage of the examined S's are P*, or *A certain S is P*. The conclusion may be: *Every S is P*, *A certain percentage of S's are P*, or *A certain other* (e.g. *the next*) *S will be P*.

Now every attempt to justify these inferences by the principles of deductive logic has conspicuously failed. Each of these inferences involves an obvious formal fallacy. In every case there is reference in the conclusion to something which is neither 'contained in', nor entailed by, anything in the premisses. The arguments may be stated either in the form:

Some A's (viz. the examined) are B,
∴ Every A is B,

or in the form:

If anything is P it is Q,
Something is Q,
∴ It is P.

The former mode of statement involves an 'illicit process'; the latter involves the fallacy of 'affirming the consequent'. It is now fairly generally agreed that in order to justify these inferences, the principles of deduction must be supplemented, and supplemented in two very different ways.

They must be supplemented in the first place by the general principles of probability, in virtue of which the conclusion will assume the form '*Therefore every A is probably B*', or '*Therefore probably P*'. In this way the formal fallacy is removed.

Secondly, they must be supplemented by some special assumption concerning the structure of the world to which our reasoning is applied.

These two kinds of extension in the theory of inference differ in certain important ways. The principles of probability are on the same footing as the principles of deduction, both in their status and in the way in which they are known. Like the principles of deduction they belong to 'pure Logic'. In the realm of pure logic it is possible to exclude all reference to the distinctive characteristics of the world in which we happen to live, and with which our reasonings are in the main concerned. The pure logician likes to describe his science as one concerned with what is true of every possible world, or as one concerned with principles of inference which are valid whatever may be the material to which they are applied. The axioms of probability conform to this description. They are also on the same basis as the axioms of deductions in respect of the manner in which they may be known.

Postulates concerning the structure of this particular world are in a very different position. They concern what is actually the case as opposed to what is merely possible; and they are not established by the procedure of intuitive induction.

The central problem of induction thus assumes a three-fold character:

(1) To formulate with precision the essential assumptions required concerning the structure of the world if inductions are to be established. This we may call 'the *ontological* problem',[1] and the assumptions themselves 'the ontological postulates'.

(2) To exhibit how from these assumptions and the general principles of pure logic (including therein the principles of probability) the validity of induction is established. This is the specifically *logical* problem.

(3) To determine the manner in which the special assumptions are known, or the grounds on which their acceptance is reasonable. This is the *epistemological* problem.

It will be convenient to discuss first the purely logical problem.

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[1] Ontology being the special branch of philosophy concerned with the general characteristics of whatever exists or is actual.

In principle, there is no reason why the logician should not be content to concern himself with this second problem alone, examining such assumptions as may appear interesting and leaving it to others to decide whether such assumptions are in point of fact true; and how, if true, they are known. In fact, this is all that the 'pure' logician is concerned with. His position is somewhat analogous to that of the geometer who deduces the consequences of any set of primitive propositions concerning the constitution of a certain kind of order, leaving it to others to decide whether the characteristics of any given system are realized by the spatial order of the actual world.

But the analogy is imperfect. It is part of the admitted function of a logician to formulate general principles in accordance with which assumptions hypothetically entertained may be verified. By implication, therefore, he may hope to contribute something to the solution of the problem under discussion. More important, perhaps, are the general reasons which may be adduced against a too rigid separation of the philosophical sciences. In the present case the three parts of the problem are not wholly mutually independent. In the examination of the principles of probable inference greater interest attaches to certain special cases, viz. to cases in which these inferences are practically required. So, too, in the selection of ontological postulates. The objective here is to make the minimum of dubious assumption, to ascertain the least that is required to enable inference to proceed.

The postulates with regard to the structure of the world must conform to the logical conditions. They must enable the calculus of probabilities to be fruitfully employed. The probabilities of the conclusions must be relatively determinate and, if we are not to rest in virtual scepticism, they must be relatively high.

Determinate conclusions require determinate premisses. The most obvious way in which determinate probabilities are obtained requires premisses in which determinate probabilities are assigned to a set of alternatives which are mutually exclusive and collectively exhaustive of the possibilities in the case. We require to know, that is to say, that the probability of either a or b or c . . . or $n = 1$. The simplest case is that in which there is no *a priori* reason, and no empirical reason, for supposing that one of these alternatives is more likely to be true than any other. In such cases the alternatives are equally probable until special evidence to the contrary is adduced. These conditions are realized in the elementary problems which relate to the tossing of a coin or a die. That a penny will fall ' heads ' and that it will fall ' tails ' are equally probable, and that one or other will be uppermost is certain. The certainty that a die will fall with one of its six faces uppermost is distributed into six equiprobabilities among the possible alternatives.

The simplest way in which probabilities are enhanced is by the elimination of alternatives. If any alternative is ruled out by special evidence, its initial probability is distributed among the remainder. High probabilities are obtained chiefly in two ways; by a severe restriction in the number of alternatives that remain, and by special evidence of ' loading '. The smaller the number of equiprobable alternatives the greater the probability of each. The chance that a penny will fall head uppermost is greater than the chance that a die will fall six uppermost. The simplest case is that in which there are only two equiprobable possibilities to be considered. It has sometimes been supposed that these conditions obtain whenever we know that something is either p or non-p. This, however, is clearly not the case since, in general, there are many more ways of being non-p than there are ways of being p. Being non-p is itself a set of alternatives. The alternatives p or not-p are equiprobable only when they express the two halves in a range of determinates under a given determinable. We know, for example, that every man has stature. If the question be whether he is tall or not the antecedent probabilities are equal. He must be above the average height or not, and both are equally likely. Any special evidence in favour of one of these alternatives will yield a measure of probability greater than one in two.

The conception of ' loading ' may be generalized in the theory of probability to cover all cases in which the total probability of a set of alternatives is unequally distributed among the alternants. Commonly, knowledge of loading is provided only by special evidence. Equiprobabilities may issue out of ignorance, unequal probabilities are dependent upon special knowledge. But such knowledge may be provided by analysis of the properties involved. If we know only that an object is coloured the determinate shades of colour it might possess are equally probable. But if we know that it is either pure red or orange, the two alternatives are not equally probable. Pure red is approximately specific, whereas 'orange' covers a wide range of hues. Hence the antecedent probability that the object possesses one or other of the latter is higher than the probability that it possesses the former.

Such, in general, would seem to be the kind of logical conditions to which the ontological postulates must conform.

§ 2. Ontological Conditions: Determinism and Simplicity.

A world which allows inductive generalization, it would seem, must exhibit two characteristics; some measure of determinism and some limitation in complexity. It has long been held that the validity of induction depends upon principles of causal determination. The doctrine of Universal Causation was fundamental to Mill, and in some form it has appeared in most expositions of inductive theory. It has been less widely acknowledged that the law of Causation, in the forms in which it is commonly stated, is neither necessary nor sufficient for the purpose it is intended to fulfil.

The term 'cause'—in any natural usage—has an application which is restricted in certain definite ways. It applies most naturally to events, or to objects in a spatio-temporal order. It does not apply to systems of 'universals', e.g. to anything like the systems of numbers. It has, however, been argued by Mr. Keynes that we require a theory of induction which will apply beyond the realm of purely empirical data. If purely inductive arguments give any plausibility to a generalization concerning heat and friction it is difficult to deny that they may give a similar plausibility to generalizations concerning numbers and angles. By purely inductive reasoning a child may be led to see that all the angles which may be inscribed in a semicircle are right angles. Many important mathematical theorems have in fact been discovered in this way. It would be paradoxical to argue that a belief so attained is wholly unreasonable, and equally paradoxical to assume that it depends upon the Law of Universal Causation.

The point would seem to be that causation is a species of a certain general type of connection every form of which can properly be described as a case of determination. The general assumption required is that properties are determined in a systematic way. The essential property of determination is that it yields propositions of the form: *If p is the case then q is the case as well.* Of such relations the clearest is that of logical entailment. When a property p entails a property q we can, in general, immediately assert that if anything is p then it is also q. In the simplest cases of this kind the procedure of induction is not so much invalid as wholly superfluous. From a single case we may see that, inasmuch as q is entailed by p, anything that is p is also q. But sometimes relations of entailment are extremely complicated. A long and involved chain of reasoning may be required to exhibit the necessity of the connection. In such cases induction anticipates formal proof, and in fact provides the motive for discovery of the proof. The fact that in these cases induction is ultimately displaced by deduction perhaps explains, in large measure, the attempt to exhibit every inductive generalization merely as an anticipation of a necessary law.

In the most general sense of the term, determination is exhibited by any system involving elements some of which are primary and others derivative. Where the elements are characters we may speak of the primary elements as '*generators*' and the derivative characters simply as '*derivatives*'.

Such a system can be conceived in which some groups of characters are determined by each of a set of generating characters acting in isolation, other groups by two generators acting in conjunction, others by three and so on. Relations of this sort in a system of characters would result in a world possessing 'natural kinds'. Objects would fall into classes which were distinguished from one another by a multiplicity of characters, and small sets of characters in an object would be the sign of a larger group of characters. There would thus be true propositions of the form

If *abc* then *pqr*.

This, we have seen, is the kind of conclusion to which induction commonly leads.

Various complications might be introduced into the conception of a system. It might be logically impossible for certain generators to be combined, even though their derivatives be mutually compatible. We can conceive that the derivatives of characters acting in conjunction would differ from those which issue from the same characters acting in isolation. We can either admit alternative generators for the same character, or suppose that each character is generated in only one way. In general, such complications will result from the combination of four fundamental possibilities. They arise in so far as we admit that both generators and derivatives may be alternative or conjunctive.

I. **Alternative Generators.**—A single derivative d_1 may be determined either by g_1 *or* g_2 *or* g_3.
II. **Conjunctive Generators.**—A single derivative requires the conjunction of generators g_1 *and* g_2 *and* g_3.
III. **Alternative Derivatives.**—A single generator may determine *either* d_1 *or* d_2 *or* d_3.
IV **Conjunctive Derivatives.**—A single generator determines the conjunction d_1 *and* d_2 *and* d_3.

If the possibility of alternative generators alone be admitted we can infer: *If* g_1 *then* d_1 or *If* g_2 *then* d_1 or *If* g_3 *then* d_1. We cannot infer *If* d_1 *then* g_1 or *If* d_1 *then* g_2, etc., we can infer only: *If* d_1 *then* g_1 *or* g_2 *or* g_3.

If the possibility of alternative derivatives alone is admitted we can infer *If* d_1 *then* g_1 or *If* d_2 *then* g_1 or *If* d_3 *then* g_1, but we cannot infer *If* g_1 *then* d_1, etc. We obtain again only an alternative consequent: *If* g_1 *then either* d_1 *or* d_2 *or* d_3. It is obvious that if we admit both possibilities our inferences become very much more complex. With any considerable increase in the number of alternative generators and derivatives they become practically unmanageable.

The distinction between generators and derivatives, it will be seen, is of logical importance only in so far as some of these possibilities are involved.[1] In the simplest case, in which a single generator determines only a single derivative, we may infer from a single generator to a single derivative and *vice versa*; *If* g_1 *then* d_1 and *If* d_1 *then* g_1. It then becomes a matter of no logical importance which we describe as generator and which as derivative. The most common assumption is that we may have alternative generation but not alternative derivatives. Conjunctive generators and conjunctive derivatives are readily admitted. The admission of conjunctive derivatives mitigates the difficulties that arise from alternative generators. If d_3 is produced by either g_1 or g_2, but g_1 and g_2 differ in their other derivatives, we can select the appropriate derivative in any given case by attending to the concomitants of the derivative with which we are concerned.

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[1] The possibility of alternative derivatives is commonly excluded. This, perhaps, follows from the definition of a 'generator'. But for a general survey of the possibilities this case must be admitted.

Thus if g_1 determines d_1, d_2, d_3
and g_2 ,, d_3, d_4, d_5

we obtain the generalizations :

If d_1, d_2, d_3 then g_1
If d_3, d_4, d_5 then g_2.

The common derivative d_3 is referred to g_1 or to g_2 according as it is accompanied by d_1 and d_2 or by d_4 and d_5.

As Dr. Stebbing remarks,[1] ' the procedure of a coroner's court is based upon the assumption that if the total characters of the effect-occurrence, viz. the death of the person, be made determinate, then the precise character of the cause-occurrence can be ascertained '. Conversely, it is assumed by thoroughgoing determinists that if the total nature and circumstances of a man were known we could predict whether he was going to commit a murder. That is, from a complete knowledge of the generator we can infer all derivatives.

In general, there may be systems in which the ultimate generalization would require a conjunction both in the antecedents and in the consequents, i.e. laws of the form :

$$\text{If } (g_1, g_2, \ldots g_n) \text{ then } (d_1, d_2, \ldots d_n).$$

Any decrease in comprehensiveness of the antecedent would require an alternative consequent, and any decrease in the comprehensiveness of the consequent would require an alternative antecedent.

It is easy to see how systems of this type may provide material for the calculation of probabilities. Given that a character is determined by one of a set of possible generators or determines a set of possible derivatives we can secure the fundamental set of equiprobable possibilities from which inductive reasoning may commence. Thus in one of the simplest cases we may apply the principle of independence. If both Cv and $C\bar{v}$ are followed by P we may infer that P depends upon one of the conjunctive characters comprised within C. The initial probability that P depends on a given factor in C, say *a*, will vary with the number of equiprobable components C contains. Its final probability will depend upon the degree to which some of these admit of elimination and upon the evidence for ' loading '. But the same considerations enable us to see that the bare principle of determination is insufficient. The simple symbolism employed obscures the possibly indefinite complexity of which C may admit. No determinate initial probability is given unless limits to this complexity are set ; and unless the limits are narrow the probabilities attained will be extremely small. For this reason, a postulate of simplicity is required. With special reference to the argument from Analogy, Mr. Keynes has suggested a postulate described as *the Limitation of Independent Variety.*

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[1] ' Modern Introduction to Logic ', p. 263.

' As a logical foundation for Analogy, therefore, we seem to need some such assumption as that the amount of variety in the universe is limited in such a way that there is no one object so complex that its qualities fall into an infinite number of independent groups (i.e. groups which might exist independently as well as in conjunction) ; or rather that none of the objects about which we generalize are as complex as this ; or at least that, though some objects may be infinitely complex, we sometimes have a finite probability that an object about which we seek to generalize is not infinitely complex '.[1]

The types of system so far considered have been conceived primarily in relation to the requirements of indirect induction and of arguments from analogy. Here the evidence consists of observed conjunctions in which the characters involved are not intrinsically related and are merely present or absent. Now the type of system which needs to be postulated may possibly differ according to the type of inductive inference we wish to justify. We may, therefore, briefly consider the sort of system which seems to be presupposed in the process of universalization, i.e. in the formulation of laws of functional dependence.

In this case the type of determination involved would seem to differ from that which is appropriate to characters which are merely present or absent. Functional determination is a relation between determinables. One determinable may be said to be wholly independent of another when each may vary throughout the whole range of its determinates without involving any variation in the other. When there is not this perfect independence one is at least partially dependent on the other. Dependence may thus vary in degree. The least that it entails is that there are *some* variations in the one determinable which involve variations in the other.

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[1] J. M. Keynes, ' A Treatise on Probability ', p. 258.

A set of determinable characters in which each was wholly independent of all the rest would not constitute a system. A system arises when the members of the set are restricted in their possibilities of independent variation. To justify the procedure of functional induction we may thus postulate a special principle which may be described as the *limited possibility of independent variation.* We may suppose that each of the determinables in terms of which the actual world may be described is subject to such restrictions. Their limitations in respect of independent variation may assume two principal forms:

(1) A character may be limited in respect of the *number* of other characters of which it is independent. That is to say, in the case of every character there is at least one other upon which it is functionally dependent.

(2) A character may be limited in respect of the *type and range of freedom* it possesses in relation to the characters upon which it is in part, at least, dependent.

These restrictions admit various possibilities between the extreme case in which every character is wholly independent of every other and the other extreme case in which every character is wholly dependent upon every other. A world in which every variation in every character involved a variation in every other character would constitute a rigidly deterministic system. No change could occur in any character of any one thing without involving a change in every character of every other thing. Relaxing this condition, we obtain various possible systems which are more or less ‘ loosely jointed ’. To all appearances the actual world would seem to be such a loosely jointed system.

Taking the general type of determining relation in the system to be represented by the formula

$$P = f(ABC \ldots N),$$

the first limitation restricts the number of 'arguments' in ABC . . . N; the second imposes a limitation upon the complexity of the function *f*. Here, too, the required ontological postulate involves not only determination, but also some restriction upon the complexity of the system presupposed. We cannot admit laws of functional dependence which require an infinite number of variables or infinitely complex functional relations.

§ 3. The Epistemological Problem.

Concerning our knowledge of the ontological postulates, three views may conveniently be distinguished; (*a*) that they possess the status of hypotheses, (*b*) that they are 'regulative principles', (*c*) that their truth is known in some way directly, or, at least, in a non-inferential way.

In the most general sense of the term an 'hypothesis' is a proposition 'hypothetically entertained', i.e. merely supposed for the purposes of examination. Considered *qua* 'hypothesis', there is no reason whatever for supposing it to be true. It may, however, be 'verified'. If it be subsequently discovered that the proposition is logically entailed by something already known verification amounts to demonstration. It is agreed, however, that the ontological postulates are not verified in this way.

The alternative is inductive verification. The hypothetical method of induction assumes the following form. A generalization is proposed, i.e. hypothetically entertained. Next, its consequents are deductively ascertained. These will comprise a number of propositions the truth or falsity of which may be directly ascertained. If any prove to be false the hypothesis is disproved. If all are found to accord with fact the probability of the hypothesis is enhanced. Symbolizing the hypothesis by h and the consequences by $c_1, c_2 \ldots c_n$ the procedure conforms to the following scheme:

If h then $c_1, c_2 \ldots c_n$,
c_1, c_2, c_3, etc., are in fact the case,
$\therefore$ h is probably true.

The final determinate probability of h depends upon its antecedent probability, upon the number and the variety of the consequences which are found to be true, and the extent to which they eliminate alternative hypotheses. But the validity of the method itself depends upon the ontological postulates. To rely exclusively on this method to establish the postulates would therefore constitute a circular argument. The most that can be said for this procedure has been said by Mr. Keynes, who points out that if there be some alternative ground which yield an antecedent probability to the postulate the hypothetical method may be employed to enhance this probability. But clearly this alternative ground cannot consist in a further hypothesis of a similar kind.

It is sometimes said that science depends upon principles of two fundamentally different kinds. It depends, firstly, upon principles which assert, hypothetically or categorically, that something is the case; it depends, secondly, upon certain 'demands' or 'imperatives' proclaimed by the thinker. Accounts of the second class of principles, sometimes described as 'regulative' or 'methodological' principles, assume very different forms, and their status is often exceedingly obscure.

The conception of a 'demand' is implicit in that of the 'postulate' in one sense of this term. By a postulate we may mean something that a thinker demands to be granted at the commencement of a proof. Such demands are not unreasonable since a proof must start from something. If nothing be granted, a proof cannot begin. The demand in such cases may be of a purely *ad hominem* nature. The reasoner is prepared to start from different points in accordance with what the doubter is willing to accept. The term may also be applied to such demands as may be made of every reasonable inquirer. As such it may be interpreted as embracing only what is actually self-evident or is a matter of undeniable fact. The conception cannot, however, be generalized to the point at which the demand is issued to Nature itself. There is no guarantee that Nature would respond in the appropriate way.

In another sense the postulate is a statement of some requirement beyond the principles of logic and matters of established fact. As such there is nothing in the definition of the case which can throw any light upon its epistemological status.

A third conception is that of Professor L. J. Russell who refers to certain demands which 'seem to express something in man's attitude to the facts rather than to express the mere facts themselves '.[1] Such demands are demands for simplicity, of which, perhaps, the typical example is the familiar Occam's Razor—the principle that *entities must not be multiplied beyond necessity*. Clearly this demand cannot be addressed to the facts themselves. It is obviously the scientist, rather than Nature, who should practise economy. If, however, this means that the scientist must not assert that anything exists in the absence of evidence for its existence the principle, though certainly true, is extremely trite. The rule is primarily applicable to the construction of hypotheses. As such it does not follow from any truth concerning the nature of the material with which the thinker is concerned, nor from any truth concerning his own nature. It is not so much an expression of an ' attitude of mind ' as an expression of a specific intention—the intention to discover a *simple explanation.*

This intention ' regulates ' scientific thought in all its phases. The scientist seeks a simple order in nature ; he seeks simple laws and simple hypotheses. The simplest order is not truer than orders that are more complex. It merely happens to be the order which is being sought. There may be more complicated orders but the scientist is not interested in these. We may classify objects on various principles. On one principle we obtain simple laws. On a different principle more complicated laws would result. Both are equally true. The demand is not that the facts should be simple but that the *formulation* should be the simplest of which the facts admit.

If a distinction is to be drawn, as indeed it must, between ontological postulates and regulative principles, it is convenient to define the latter as those that follow from the nature of the inquiry as contrasted with those that follow from the nature of what the inquiry is about. But so defined it is clear that the regulative principle can throw no light upon the *validity* of an inductive inference. It remains, therefore, to inquire whether the truth of the ontological postulate may be known in some more direct way.

It is evident that some forms of the doctrine of determinism are less improbable than others. We have contemplated systems of various types in which some characters are supposed to determine others. The form of such systems will vary according as we suppose that the simple characters of the system are determined by other simple characters alone, or by some conjunction of characters, and according as they are determined in a single way or admit of alternative modes of determination. These characters may also be conceived either as themselves determining simple characters or as determining a certain conjunctive set; and they may be conceived as determining only one such character or set; or as determining a group of alternatives. It is clearly less improbable to suppose that a simple character is determined by some or other character or conjunction of characters than it is to suppose that simple characters are determined only by simple characters. And it is less improbable to suppose that a character may be determined in at least one way than to suppose that it is determined in one and only one way.

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[1] *Proc. Arist. Soc., N.S.*, XXV., ' Science and Philosophy '. A discussion of this view is also to be found in Dr. Stebbing's ' Modern Introduction to Logic ', Ch. XXI.

The least extreme postulate of determinism (and by the least extreme we mean the least improbable) would seem to be the following. We might assert of the characters within a given system, that each one is determined. It is determined either by some simple character, or by some conjunction of characters or by some set of alternative simple or compound characters. If, further, it be granted that no character can occur in isolation (i.e. that nothing possesses only a single character) this postulate of determinism will then be necessarily true. The two statements, in fact, turn out to be equivalent. The one says that no character can occur alone, the other asserts that whenever any character occurs it must be accompanied by some other character or collocation of characters comprised within the system.[1] This proposition would be consistent with a state of affairs in which the conjunctions were formed 'by chance'. Clearly, then, in spite of its self-evidence the proposition is devoid of value. The postulate of determinism is then in itself useless. It becomes important only if it is possible in some way to restrict the range of alternative causes or effects. But directly such restrictions are introduced self-evidence disappears.

One expedient by which this restriction might conceivably be secured is by direct judgments of determination and direct judgments of irrelevance. If we cannot directly apprehend the necessity for the required general principle of determination we may nevertheless be able to apprehend special cases of determination and special cases of irrelevance.

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[1] This curious consequence is the result of interpreting 'determinism' strictly in terms of the relation in virtue of which we are entitled to assert If *p* then *q*.

It has been seen that certain arguments may be expressed in the form: *If a set of instances of P agree only in the presence of certain other characters, ABC . . . N, then some selection of these characters is probably invariably associated with P.* When, however, ABC . . . N is a large and indeterminate assemblage of characters the conclusion has only an insignificant measure of probability. Mr. Keynes has suggested that some of the components in such an aggregate may be immediately judged to be irrelevant.

In many cases we argue to some conclusion concerning the mental states of others from data which relate simply to mental states of our own. The validity of this argument may be challenged on the ground that the states from which I infer agree in being 'mine', and this character may be relevant to the conclusion drawn. This objection is met by Mr. Keynes by the contention that the difference is one which can be known directly to be irrelevant. It is one upon which no inferred character can possibly depend.

The judgment of irrelevance has nothing to do with the fact that a 'psychological subject' is concerned. It is a special case of what we may describe as the 'irrelevance of the particular': *If two things are only numerically distinct, i.e. if they do not differ in character, the distinction between them is irrelevant to any inductive conclusion.*

It is easy to see how such judgments of irrelevance may arise. The principle follows directly from the postulate of determinism, whatever may be the form this postulate assumes. In all its possible forms it is an assertion to the effect that *characters* are determined, and determined by *characters* alone. A purely numerical difference is therefore irrelevant, since this is not a difference in character.

But though self-evident the principle is barren, since we are never in fact concerned with things that are merely numerically distinct. What is required is a principle for determining irrelevant *characters*. At first sight, Mr. Keynes' second example of a judgment of irrelevance promises to satisfy this condition. It is suggested that among the antecedent certainties of induction is the knowledge that differences in space and time are irrelevant.

'Mere position in space or time cannot possibly affect as a determining cause any other characters.'[1] It is clear, however, that *relative* positions in space and time are not irrelevant to causal process. The positions of a match and the time at which it is struck are not irrelevant to the occurrence of an explosion. All that we are entitled to say is that *absolute* position and *absolute* time are irrelevant. But this judgment is again devoid of significance for ordinary inductive process. The things concerning which we generalize differ from the data of generalization in their relative spatio-temporal properties. With this the *a priori* judgment of irrelevance is powerless to deal. In fact, relative positions are among the more important subjects of the generalizations of physics.[2]

It remains to inquire whether the required simplicity in nature may not be ascertained by some procedure of analysis, and by turning attention from bare presence or absence to relations of functional dependence. There is much to suggest that causal determination is discerned most directly by the observation of concomitant variation; and that inference from cases of mere presence and absence is a roundabout device for ascertaining this relation from less articulate data.[3]

The conception of functional dependence between determinables, *prima facie* at least, is promising for the postulate of simplicity. The objects with which induction is concerned appear, at first sight, to possess an indefinitely great number of characters; but this complexity yields to analysis. Specific characters can be ranged in series under appropriate determinables and so ranged that the species under the same determinables are mutually incompatible. Given then a finite number of determinables—even though each comprises an infinite number of determinates—it will follow that any objects can possess only a finite number of specific characters.

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[1] Keynes, 'Treatise on Probability', p. 255.

[2] The irrelevance of absolute position (if absolute position be admitted) would seem to follow from the irrelevance of the particular. Terms like 'Here', 'Now', 'This', and 'I', when divested of relational characters, agree in expressing entities of a type fundamentally different from characters.

[3] The distinction corresponds closely to that in statistics between the theory of attributes and the theory of variables. Cf. Ch. X. *supra*, and G. Udny Yule, 'Introduction to the Theory of Statistics', Pt. I., Ch. I.

The simplification, in fact, might seem almost to prove too much. If there be, say, only a hundred determinables—and it would not be easy to enumerate many more—narrow limits will, it might seem, be set to the number of purely functional laws of which the universe admits. But in point of fact the complexity of a science does not arise merely from the number of determinable characters involved. It arises in large measure from the number of particulars to which characters are referred. Only an extremely limited number of determinables are involved in the propositions of physics. The laws relating to the gravitational behaviour of two bodies may be fairly simple; but when the number of bodies in spatio-temporal relation is appreciably increased the laws in question become exceedingly complex. It is, perhaps, only in psycho-physical studies that the number of distinct determinables involved becomes relatively large. The complexity of the universe would thus seem to be primarily a matter of extension. It is a matter of the number of individual things it contains rather than of the number of determinable characters they possess.[1]

A set of determinable characters provides appropriate material for a deterministic system. What is the evidence that the characters of the actual world conform to this description? In the complete procedure of functional induction there are three important phases of epistemological interest—the central process of universalization, the process of generalization, and the elimination of the concrete. The data of observation, we have seen, assume the following form:

$$\begin{array}{llll}
S_1 \text{ is } q_1 \text{ when } p_1 & S_2 \text{ is } q_1 \text{ when } p_1 & \ldots & S_n \text{ is } q_1 \text{ when } p_1 \\
S_1 \text{ is } q_2 \text{ when } p_2 & S_2 \text{ is } q_2 \text{ when } p_2 & \ldots & S_n \text{ is } q_2 \text{ when } p_2 \\
\cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \ldots & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
S_1 \text{ is } q_n \text{ when } p_n & S_2 \text{ is } q_n \text{ when } p_n & \ldots & S_n \text{ is } q_n \text{ when } p_n
\end{array}$$

In the simplest cases it is presumed that S remains constant in respect of the characters other than p and q. In fact, unless S were constant or its other characters irrelevant to q, there is no reason to expect that we should obtain a set of facts conforming to this scheme. When they fail to conform to it, it is frequently possible to eliminate the anomaly by a further analysis of S. Detecting a further variable we construct a similar table in a third dimension. By multiplication of the dimensions of tabulation, S is progressively divested of its content. If the process is ever complete, we obtain a purely functional formula of the form :

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[1] If the number of determinate characters were also finite a limit would then be set to the number of things that could exist—granted, that is, the principle of the 'identity of indiscernibles'.

Q is completely determined by l, m, n, o, p.

Universalization, in any one of its dimensions, is sufficiently guaranteed by the general principles of probability. We are concerned, let us say, with two series of determinates $p_0p_1p_2 \ldots p_n$ and $q_0q_1q_2 \ldots q_n$ under the determinables P and Q.[1]

If we find, in a chance selection of instances, that for every value of P there is a corresponding value of Q, the presumption that the two determinables have the required connection becomes exceedingly strong.[2] So far, at any rate, the process has an *a priori* guarantee.

When the evidence assumes a form which yields a presumption of determinism, when it suggests for example that the character p is wholly determined by a, b, c, d, e, the process of mere generalization would seem to follow from the irrelevance of the particular. The data provisionally included in the line of generalization—i.e. in the first dimension of inference—become, by the progressive analysis of S, completely resolved into a series of instances which are merely numerically distinct. Since, however, the presumption is that p varies only in virtue of variation of some other character each value in the table may be immediately generalized.[3]

The presumption of complete determinism and the consequent generalization presupposes this complete analysis of S.

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[1] The determinates, p_0 and q_0, are introduced to cover the limiting cases in which a character is absent. Cf. *supra*, p. 331.

[2] If the correspondence assumes a 'simple' form, e g. that of direct or inverse variation the probability of determinism is high even on the basis of a single instance of each value of p. But however complicated or irregular the correspondence a similar argument may be based on two instances of each value of p.

[3] Generalization, pure and simple, at this point loses all interest, since there is also a presumption that no two things are merely numerically distinct. Our prima facie generalizations are thus in effect genuine universalizations in which certain variant characters are believed to be irrelevant.

It presupposes, that is to say, that S has a finite number of determinable characters.

That the number of distinct determinables in the world is finite is a plausible proposition, but it is not an *a priori* truth. We cannot assert it merely by contemplating its terms. If we accept it, we do so on a starkly empirical basis. But, if we find it so, we do not find it in a single observation. We do not see it all at once, but we may ' come to see ' it in the course of continued observation. We come to see it much as we come to see the dimensions of an object which is larger than the field of vision.

This conclusion is subject to some residual doubts. There may be determinables which have ceased, before the advent of man, to characterize the existent ; there may be characters which have yet to appear ; there may be existent determinables outside the range of human observation. But all these possibilities may be realized without impugning the validity of our actual inductions. The incognizable determinables must either be constant or be irrelevant to the laws which are approximately ascertained. Otherwise there would not appear to be the laws. That there are such laws, and that they are in fact so simple would seem to constitute the most tangible basis of our assumption that induction is somehow valid. These laws remain, perhaps, the most significant fact in an otherwise puzzling situation.

BIBLIOGRAPHICAL NOTES.

CHAPTER XVIII.

This chapter contains only a brief and preliminary survey of the problems with which it is concerned. From this point it must be left to the student to find his own way in what is after all—and for all—in large measure *terra incognita*. To see the problem—and this is the first step—he will do well to study Hume, Hume's disciples and Hume's critics. After Hume he will not perhaps find much in British philosophy which contributes to enlightenment until the nineteenth century, and probably not until he comes to contemporary writers will he find, on the more purely logical aspects of the question, any genuine signs of progress. The chief authorities are **W. E. Johnson, Mr. J. M. Keynes** and **Dr. C. D. Broad.** Johnson's **Logic** was unfortunately left unfinished, but in the three important posthumous articles in **Mind** (Vol. XLI.) we see something of what Part IV. of the Logic would have contained. Mr. Keynes, after his brilliant excursion into the subject, has returned to economic studies. We therefore rely on Dr. Broad. In the meantime the student should acquaint himself with not only Dr. Broad's articles previously referred to (**Proceedings of the Aristotelian Society,** Vol. XXVIII., and **Mind,** XXIX., XXXI.), but also the articles entitled '**Mr. Johnson on the Logical Foundation of Science**', **Mind,** Vol. XXXIII.

In 1924 Jean Nicod (who died in the same year) presented an original contribution to the subject in **Le Probleme Logique de l'Induction** (English Translation in the **Foundations of Geometry and Induction,** London, 1930). Nicod's views are in large measure adopted by **R. M. Eaton** in his **General Logic** (who also died in the year his work was published).

Reference should also be made to the valuable posthumous papers of **F. P. Ramsey** (**The Foundations of Mathematics,** London, 1931).

The whole problem, with special reference to the contributions of Keynes and Broad, receive excellent treatment in **Dr. Stebbing's Modern Introduction to Logic** (especially Chapter XXI.). The views of these authors are also discussed by **A. D. Ritchie** in his **Scientific Method,** and again (with some reference to Nicod) in an article on '**Induction and Probability**' in **Mind,** Vol. XXXV The same volume of **Mind** contains an article by R. H. Nisbet on the '**Foundations of Probability**'.

The broader epistemological background of the subject is provided in **Professor J. Laird's Knowledge, Belief and Opinion.**

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